

Question 12

Carry out the following integrations to the answers given.

1. $\int_1^{\sqrt{2}} \frac{1}{x^2 \sqrt{4-x^2}} dx = \frac{1}{4}(\sqrt{3}-1)$, use $x = 2 \cos \theta$

Handwritten solution for Question 1:

$$\int_1^{\sqrt{2}} \frac{1}{x^2 \sqrt{4-x^2}} dx = \int_{\pi/3}^{\pi/4} \frac{-1}{4 \sin^2 \theta} d\theta = \left[\frac{1}{4} \cot \theta \right]_{\pi/3}^{\pi/4} = \frac{1}{4} \left(\cot \frac{\pi/4}{4} - \cot \frac{\pi/3}{4} \right) = \frac{1}{4} (1 - \sqrt{3})$$

2. $\int_{\sqrt{2}}^2 \frac{1}{x^2 \sqrt{x^2-1}} dx = \frac{1}{2}(\sqrt{3}-\sqrt{2})$, use $x = \sec \theta$

Handwritten solution for Question 2:

$$\int_{\sqrt{2}}^2 \frac{1}{x^2 \sqrt{x^2-1}} dx = \int_{\pi/4}^{\pi/3} \frac{-1}{\sec^2 \theta \sin \theta} d\theta = \left[\frac{1}{2} \sec \theta \right]_{\pi/4}^{\pi/3} = \frac{1}{2} (\sec \frac{\pi/3}{2} - \sec \frac{\pi/4}{2}) = \frac{1}{2} (2 - \sqrt{2})$$

3. $\int_0^1 \frac{1}{(1+x^2)^2} dx = \frac{\pi}{8} + \frac{1}{4}$, use $x = \tan \theta$

Handwritten solution for Question 3:

$$\int_0^1 \frac{1}{(1+x^2)^2} dx = \int_0^{\pi/4} \frac{-1}{2 \cos^4 \theta} d\theta = \left[\frac{1}{2} \tan \theta + \frac{1}{4} \ln |\sec \theta| \right]_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}$$

4. $\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}} dx = \frac{\pi}{6}, \text{ use } x = \frac{\sqrt{3}}{2} \sin \theta$

Handwritten solution for problem 4:

$$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{3-4\left(\frac{\sqrt{3}}{2} \sin \theta\right)^2}} \cdot \frac{\sqrt{3}}{2} \cos \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\sqrt{3}}{2} \cos \theta}{\sqrt{3-3 \sin^2 \theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\frac{\sqrt{3}}{2} \cos \theta}{\sqrt{3(1-\sin^2 \theta)}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\sqrt{3}}{2} \cos \theta}{\sqrt{3} \cos \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \left[\frac{1}{2} \theta \right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{2} \times \frac{\pi}{2} \right) - (0) = \frac{\pi}{4}$$

Boxed notes:

$$\begin{aligned} x &= \frac{\sqrt{3}}{2} \sin \theta \\ \frac{dx}{d\theta} &= \frac{\sqrt{3}}{2} \cos \theta \\ x=0, \sin \theta &= 0 \\ \theta &= 0 \\ x &= \frac{3}{4}, \frac{\sqrt{3}}{2} \sin \theta \\ \frac{3}{4} &= \frac{\sqrt{3}}{2} \sin \theta \\ \sin \theta &= \frac{3}{4} \end{aligned}$$

5. $\int_0^1 \frac{1}{(1+3x^2)^{\frac{3}{2}}} dx = \frac{1}{2}, \text{ use } x = \frac{1}{\sqrt{3}} \tan \theta$

Handwritten solution for problem 5:

$$\int_0^1 \frac{1}{(1+3x^2)^{\frac{3}{2}}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\left[1+3\left(\frac{1}{\sqrt{3}} \tan \theta\right)^2\right]^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{3}} \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\sqrt{3}} \sec^2 \theta}{\left[1+\tan^2 \theta\right]^{\frac{3}{2}}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\sqrt{3}} \sec^2 \theta}{\sec^3 \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{3} \sec \theta} d\theta$$

$$= \frac{1}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \cos \theta d\theta = \frac{1}{\sqrt{3}} \left[\sin \theta \right]_0^{\frac{\pi}{2}} = \frac{1}{\sqrt{3}} \left(\sin \frac{\pi}{2} - 0 \right) = \frac{1}{\sqrt{3}}$$

Boxed notes:

$$\begin{aligned} x &= \frac{1}{\sqrt{3}} \tan \theta \\ \frac{dx}{d\theta} &= \frac{1}{\sqrt{3}} \sec^2 \theta \\ x=0, \tan \theta &= 0 \\ \theta &= 0 \\ x &= 1, \frac{1}{\sqrt{3}} \tan \theta \\ 1 &= \frac{1}{\sqrt{3}} \tan \theta \\ \tan \theta &= \sqrt{3} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

6. $\int_0^1 \frac{1}{\sqrt{2-x^2}} dx = \frac{\pi}{4}, \text{ use } x = \sqrt{2} \sin \theta$

Handwritten solution for problem 6:

$$\int_0^1 \frac{1}{\sqrt{2-x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-(\sqrt{2} \sin \theta)^2}} \cdot \sqrt{2} \cos \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{2} \cos \theta}{\sqrt{2-2 \sin^2 \theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2} \cos \theta}{\sqrt{2 \cos^2 \theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2} \cos \theta}{\sqrt{2} \cos \theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1 d\theta = \left[\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Boxed notes:

$$\begin{aligned} x &= \sqrt{2} \sin \theta \\ \frac{dx}{d\theta} &= \sqrt{2} \cos \theta \\ x=0, \sin \theta &= 0 \\ \theta &= 0 \\ x &= 1, \sqrt{2} \sin \theta \\ 1 &= \sqrt{2} \sin \theta \\ \sin \theta &= \frac{1}{\sqrt{2}} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

7. $\int_0^{\frac{1}{2}} \frac{1}{4x^2+3} dx = \frac{\pi\sqrt{3}}{36}, \text{ use } x = \frac{\sqrt{3}}{2} \tan \theta$

Handwritten solution for problem 7:

$$\int_0^{\frac{1}{2}} \frac{1}{4x^2+3} dx = \int_0^{\frac{\pi}{4}} \frac{1}{4\left(\frac{\sqrt{3}}{2} \tan \theta\right)^2+3} \cdot \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{4\left(\frac{3}{4} \tan^2 \theta\right)+3} d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{3 \tan^2 \theta + 3} d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{3(\tan^2 \theta + 1)} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{3 \sec^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{6} d\theta = \left[\frac{\sqrt{3}}{6} \theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{3}}{6} \times \frac{\pi}{4} = \frac{\pi\sqrt{3}}{24}$$

Boxed notes:

$$\begin{aligned} x &= \frac{\sqrt{3}}{2} \tan \theta \\ \frac{dx}{d\theta} &= \frac{\sqrt{3}}{2} \sec^2 \theta \\ x=0, \tan \theta &= 0 \\ \theta &= 0 \\ x &= \frac{1}{2}, \frac{\sqrt{3}}{2} \tan \theta \\ \frac{1}{2} &= \frac{\sqrt{3}}{2} \tan \theta \\ \tan \theta &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6} \end{aligned}$$

8. $\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}} dx = \frac{\sqrt{3}}{12}, \text{ use } x = 2 \sin \theta$

Handwritten solution for problem 8:

$$\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}} dx = \int_0^{\frac{\pi}{6}} \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}} (2\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4(1-\sin^2\theta))^{\frac{3}{2}}} d\theta = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4\cos^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{8\cos^3\theta} d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4\cos^2\theta} d\theta$$

$$= \left[\frac{1}{4} \tan\theta \right]_0^{\frac{\pi}{6}} = \frac{1}{4} \tan\frac{\pi}{6} - \frac{1}{4} \tan 0 = \frac{\sqrt{3}}{12}$$

Boxed notes:

- $x = 2\sin\theta$
- $\frac{dx}{d\theta} = 2\cos\theta$
- $x=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$
- $x=1 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$

9. $\int_0^1 \frac{1}{\sqrt{4-3x^2}} dx = \frac{\pi\sqrt{3}}{9}, \text{ use } x = \frac{2}{\sqrt{3}} \sin \theta$

Handwritten solution for problem 9:

$$\int_0^1 \frac{1}{\sqrt{4-3x^2}} dx = \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4-3\left(\frac{2}{\sqrt{3}}\sin\theta\right)^2}} \times \frac{2}{\sqrt{3}}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4-4\sin^2\theta}} \times \frac{2}{\sqrt{3}}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4(1-\sin^2\theta)}} \times \frac{2}{\sqrt{3}}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2\sqrt{4}\cos\theta} \times \frac{2}{\sqrt{3}}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2\sqrt{3}} d\theta = \left[\frac{1}{2\sqrt{3}} \theta \right]_0^{\frac{\pi}{6}} = \frac{\pi}{12\sqrt{3}} = \frac{\pi\sqrt{3}}{36}$$

Boxed notes:

- $x = \frac{2}{\sqrt{3}}\sin\theta$
- $\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\cos\theta$
- $x=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$
- $x=1 \Rightarrow \sin\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}$

10. $\int_1^{\sqrt{3}} \frac{x^2}{x^2+1} dx = \sqrt{3}-1-\frac{\pi}{12}, \text{ use } x = \tan \theta$

Handwritten solution for problem 10:

$$\int_1^{\sqrt{3}} \frac{x^2}{x^2+1} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta}{1+\tan^2\theta} (\sec^2\theta d\theta)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta \sec^2\theta}{\sec^2\theta} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2\theta d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2\theta - 1) d\theta = \left[\tan\theta - \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3} \right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4} \right)$$

$$= \left(\sqrt{3} - \frac{\pi}{3} \right) - \left(1 - \frac{\pi}{4} \right) = \sqrt{3} - 1 - \frac{\pi}{12}$$

Boxed notes:

- $x = \tan\theta$
- $\frac{dx}{d\theta} = \sec^2\theta$
- $x=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{\pi}{4}$
- $x=\sqrt{3} \Rightarrow \tan\theta=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$

11. $\int_0^3 \frac{27}{(9+x^2)^2} dx = \frac{\pi}{8} + \frac{1}{4}$, use $x = 3 \tan \theta$

Handwritten solution for problem 11:

$$\int_0^3 \frac{27}{(9+x^2)^2} dx = \dots = \int_0^{\frac{\pi}{6}} \frac{27}{(9+9 \tan^2 \theta)^2} (3 \sec^2 \theta) d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{27 \sec^2 \theta}{(9(1+\tan^2 \theta))^2} d\theta = \int_0^{\frac{\pi}{6}} \frac{27 \sec^2 \theta}{81 \sec^4 \theta} d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{3} \cos^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{6}} = \left(\frac{\pi}{12} + \frac{1}{4} \right) - \left(0 + 0 \right)$$

$$= \frac{\pi}{12} + \frac{1}{4}$$

Boxed notes on the right:

- $x = 3 \tan \theta$
- $\frac{dx}{d\theta} = 3 \sec^2 \theta$
- $\Delta x = 3 \sec^2 \theta d\theta$
- $\Delta = 0, \theta = 0$
- $\Delta = 3, \theta = \frac{\pi}{6}$

12. $\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x} dx = \sqrt{3} - 1 - \frac{\pi}{12}$, use $x = \operatorname{cosec} \theta$

Handwritten solution for problem 12:

$$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\operatorname{cosec}^2 \theta - 1}}{\operatorname{cosec} \theta} (-\operatorname{cosec} \theta \cot \theta) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cot^2 \theta}}{\operatorname{cosec} \theta} (-\operatorname{cosec} \theta \cot \theta) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} -\cot^2 \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 - \cot^2 \theta) d\theta$$

$$= \left[\theta - \cot \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - 1 \right) - \left(\frac{\pi}{6} - \sqrt{3} \right)$$

$$= \frac{\pi}{4} - 1 - \frac{\pi}{6} + \sqrt{3} = \sqrt{3} - 1 - \frac{\pi}{12}$$

Boxed notes on the right:

- $x = \operatorname{cosec} \theta$
- $\frac{dx}{d\theta} = -\operatorname{cosec} \theta \cot \theta$
- $\Delta x = -\operatorname{cosec} \theta \cot \theta d\theta$
- $\Delta = 2, \frac{\pi}{6} = \operatorname{cosec} \theta$
- $\Delta = \sqrt{2}, \frac{\pi}{4} = \operatorname{cosec} \theta$
- $\Delta = \sqrt{2}, \theta = \frac{\pi}{4}$
- $\Delta = 2, \theta = \frac{\pi}{6}$