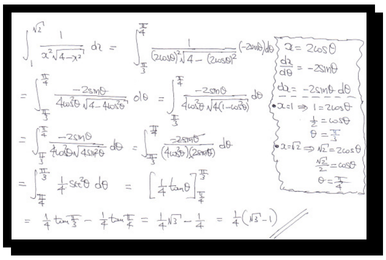
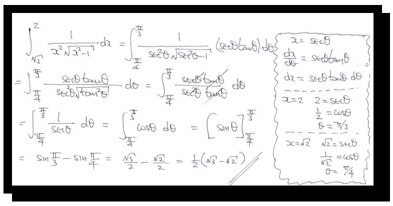
Question 12

Carry out the following integrations to the answers given.

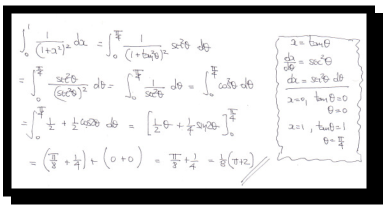
1. $\displaystyle\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\ dx = \frac{1}{4}\left(\sqrt{3}-1\right)$, use $x = 2\cos\theta$



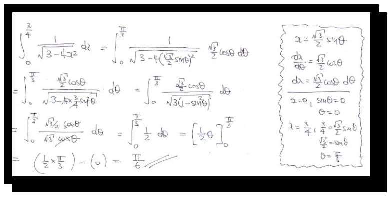
2. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\ dx = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$, use $x = \sec\theta$



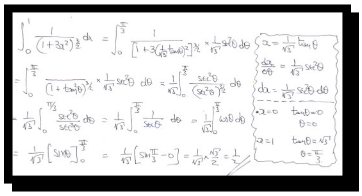
3. $\displaystyle\int_{0}^{1} \frac{1}{\left(1+x^2\right)^2}\ dx = \frac{\pi}{8}+\frac{1}{4}$, use $x = \tan\theta$



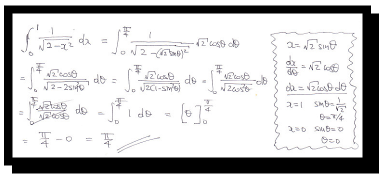
4. $$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\ dx = \frac{\pi}{6}, \quad \text{use } x = \frac{\sqrt{3}}{2}\sin\theta$$



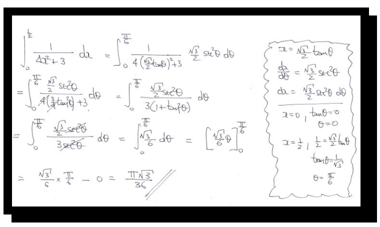
5. $$\int_0^{1} \frac{1}{\left(1+3x^2\right)^{\frac{3}{2}}}\ dx = \frac{1}{2}, \quad \text{use } x = \frac{1}{\sqrt{3}}\tan\theta$$



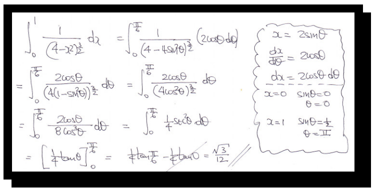
6. $$\int_0^{1} \frac{1}{\sqrt{2-x^2}}\ dx = \frac{\pi}{4}, \text{ use } x = \sqrt{2}\sin\theta$$



7. $$\int_0^{\frac{1}{2}} \frac{1}{4x^2+3}\ dx = \frac{\pi\sqrt{3}}{36}, \quad \text{use } x = \frac{\sqrt{3}}{2}\tan\theta$$



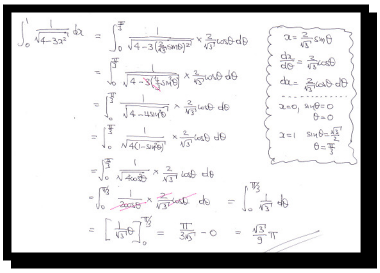
8. $\displaystyle\int_0^1 \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\,dx = \frac{\sqrt{3}}{12}$, use $x = 2\sin\theta$



$$\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}}(2\cos\theta\,d\theta)$$
$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4(1-\sin^2\theta))^{\frac{3}{2}}}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\sec^2\theta\,d\theta$$
$$= \left[\frac{1}{4}\tan\theta\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\tan\frac{\pi}{6} - \frac{1}{4}\tan 0 = \frac{\sqrt{3}}{12}$$

$x = 2\sin\theta$
$\frac{dx}{d\theta} = 2\cos\theta$
$dx = 2\cos\theta\,d\theta$
$x=0 \quad \sin\theta = 0, \ \theta = 0$
$x=1 \quad \sin\theta = \frac{1}{2}, \ \theta = \frac{\pi}{6}$

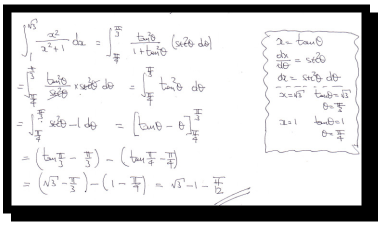
9. $\displaystyle\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \frac{\pi\sqrt{3}}{9}$, use $x = \frac{2}{\sqrt{3}}\sin\theta$



$$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-3\left(\frac{2}{\sqrt{3}}\sin\theta\right)^2}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-3\left(\frac{4}{3}\sin^2\theta\right)}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-4\sin^2\theta}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4(1-\sin^2\theta)}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4\cos^2\theta}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\pi/3} \frac{1}{\cancel{2\cos\theta}} \times \frac{\cancel{2}}{\sqrt{3}}\cancel{\cos\theta}\,d\theta = \int_0^{\pi/3} \frac{1}{\sqrt{3}}\,d\theta$$
$$= \left[\frac{1}{\sqrt{3}}\theta\right]_0^{\pi/3} = \frac{\pi}{3\sqrt{3}} - 0 = \frac{\sqrt{3}}{9}\pi$$

$x = \frac{2}{\sqrt{3}}\sin\theta$
$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\cos\theta$
$dx = \frac{2}{\sqrt{3}}\cos\theta\,d\theta$
$x=0, \quad \sin\theta = 0, \ \theta = 0$
$x=1 \quad \sin\theta = \frac{\sqrt{3}}{2}, \ \theta = \frac{\pi}{3}$

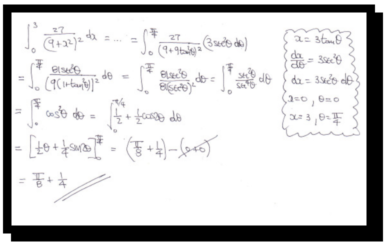
10. $\displaystyle\int_1^{\sqrt{3}} \frac{x^2}{x^2+1}\,dx = \sqrt{3} - 1 - \frac{\pi}{12}$, use $x = \tan\theta$



$$\int_1^{\sqrt{3}} \frac{x^2}{x^2+1}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta}{1+\tan^2\theta}(\sec^2\theta\,d\theta)$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta}{\cancel{\sec^2\theta}} \times \cancel{\sec^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta - 1\,d\theta = \left[\tan\theta - \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right)$$
$$= \left(\sqrt{3} - \frac{\pi}{3}\right) - \left(1 - \frac{\pi}{4}\right) = \sqrt{3} - 1 - \frac{\pi}{12}$$

$x = \tan\theta$
$\frac{dx}{d\theta} = \sec^2\theta$
$dx = \sec^2\theta\,d\theta$
$x=\sqrt{3} \quad \tan\theta = \sqrt{3}, \ \theta = \frac{\pi}{3}$
$x=1 \quad \tan\theta = 1, \ \theta = \frac{\pi}{4}$

11. $\displaystyle\int_0^3 \frac{27}{\left(9+x^2\right)^2}\,dx = \frac{\pi}{8}+\frac{1}{4}$, use $x = 3\tan\theta$



$$\int_0^3 \frac{27}{(9+x^2)^2}\,dx = \dots = \int_0^{\frac{\pi}{4}} \frac{27}{(9+9\tan^2\theta)^2}\left(3\sec^2\theta\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{\left[9(1+\tan^2\theta)\right]^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{81(\sec^2\theta)^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$$

$$= \left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{8}+\frac{1}{4}\right)-\cancel{(0+0)}$$

$$= \frac{\pi}{8}+\frac{1}{4}$$

$x = 3\tan\theta$

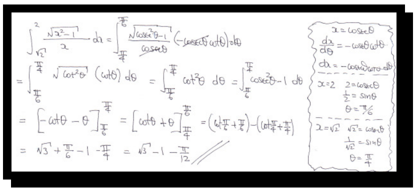
$\frac{dx}{d\theta} = 3\sec^2\theta$

$dx = 3\sec^2\theta\,d\theta$

$x=0,\ \theta=0$

$x=3,\ \theta=\frac{\pi}{4}$

12. $\displaystyle\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{3}-1-\frac{\pi}{12}$, use $x = \operatorname{cosec}\theta$



$$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\cancel{\operatorname{cosec}\theta}}\left(-\cancel{\operatorname{cosec}\theta}\cot\theta\right)d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{\cot^2\theta}\,(\cot\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^2\theta\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta-1\,d\theta$$

$$= \left[-\cot\theta-\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \left[\cot\theta+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}} = \left(\cot\frac{\pi}{6}+\frac{\pi}{6}\right)-\left(\cot\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= \sqrt{3}+\frac{\pi}{6}-1-\frac{\pi}{4} = \sqrt{3}-1-\frac{\pi}{12}$$

$x = \operatorname{cosec}\theta$

$\frac{dx}{d\theta} = -\operatorname{cosec}\theta\cot\theta$

$dx = -\operatorname{cosec}\theta\cot\theta\,d\theta$

$x=2 \quad 2=\operatorname{cosec}\theta \quad \frac{1}{2}=\sin\theta \quad \theta=\pi/6$

$x=\sqrt{2} \quad \sqrt{2}=\operatorname{cosec}\theta \quad \frac{1}{\sqrt{2}}=\sin\theta \quad \theta=\frac{\pi}{4}$