$$Let \quad I=\int_0^{3\pi/4}\frac{1}{\sqrt{2}+\cos x}dx \; = \; \frac{1}{\sqrt{2}}\int_0^{3\pi/4}\frac{1}{1+\frac{1}{\sqrt{2}}\cos x}dx$$

Use a $=-\dfrac{1}{\sqrt{2}}$

$$I=\frac{1}{\sqrt{2}}[2(\frac{1}{\sqrt{1-a^2}})\arctan\sqrt{(\frac{1+a}{1-a})}\tan\frac{3\pi}{8}]$$

$$=(\frac{1}{\sqrt{2}})(2)(\frac{1}{\sqrt{½}})\arctan\sqrt{(\frac{1-\sqrt{½}}{1+\sqrt{½}})}\tan\frac{3\pi}{8} \; = \; 2\arctan\sqrt{\frac{(1-\sqrt{½})(1+\sqrt{½})}{(1+\sqrt{½})^2}}\tan\frac{3\pi}{8}$$

$$=2\arctan\frac{1}{\sqrt{2}+1}\tan\frac{3\pi}{8}$$

$$\tan\frac{3\pi}{8}=\tan(\frac{\pi}{2}-\frac{\pi}{8})=\cot(\frac{\pi}{8})$$

$$\tan\pi/4=1=\frac{2\tan\pi/8}{1-\tan^2\pi/8}$$

Quadratic formula gives $\quad \tan\pi/8=\sqrt{2}-1$

$$I=2\arctan(\frac{1}{\sqrt{2}+1}\frac{1}{\sqrt{2}-1})=2\arctan 1$$