$$P(x)=(x-a)^m Q(x)$$

$$P`(x), \text{ using the product rule, } =mQ(x)(x-a)^{m-1}+(x-a)^m Q`(x)$$
$$=(x-a)^{m-1}(mQ(x)+(x-a)Q`(x))$$

The bit in the second bracket is a polynomial as it's just products and sums of polynomials

There will always be a factor of $(x-a)$ in the derivatives, provided the polynomial is differentiated fewer than 'm' times

$$I=\int_{-1}^{1} x^m P_n(x)\,dx$$
$$u=x^m \rightarrow u`=mx^{m-1}$$
$$v=\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \leftarrow v`=P_n(x)$$

$$I=[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n]_{-1}^{1}-m\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

The bit in the square brackets is zero. The first part of the question shows that (x^2-1) will be a factor of all derivatives, and substituting -1 or 1 into that factor gives zero.

$$I=0-m\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$u=x^{m-1} \rightarrow u`=(m-1)x^{m-1}$$
$$v=\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n \leftarrow v`=\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$$

$$I=0-m[x^{m-1}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n]_{-1}^{1}+m(m-1)\int_{-1}^{1} x^{m-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$$

The square bracket is zero again, for the same reason.

Eventually we are left with:

$$\pm m!\int_{-1}^{1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx=\pm m![\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n]_{-1}^{1}=0$$

By the above logic, or if $n=m+1$ then it's obviously 0 anyway

If we use $n=m$ then we should eventually arrive at:

$$I=(-1)^n n!\int_{-1}^{1}(x^2-1)^n\,dx$$ (If you write the first few by parts things you see that the sign on the integral alternates each time)

Let us investigate $\int_{-1}^{1}(x^2-1)^n\,dx$

It's even: $=2\int_{0}^{1}(x^2-1)^n\,dx$

Use $x=\sin t$

$$dx=\cos t\,dt$$
$$x^2-1=-\cos^2 t$$

Limits: $\arcsin 1=\pi/2, \arcsin 0=0$

Integral becomes $2\int_{0}^{\pi/2}(-1)^n\cos^{(2n+1)}t\,dt=2(-1)^n\frac{(2^{2n})(n!)^2}{(2n+1)!}$

So I $=\frac{(2^{2n+1})(n!)^3}{(2n+1)!}$??