Particle Mechanics

Basic Concepts

The DISPLACEMENT of a particle is the position vector of its final position referred to its initial position as origin and hence requires both a magnitude and a direction for its description (i.e. it is a VECTOR quantity). The basic unit of measurement is the metre (m). It is very important to distinguish between the distance moved in a given time interval and the magnitude of the displacement.

The VELOCITY of a particle is defined as the rate of change of displacement and hence is also a vector quantity. Be careful not to confuse velocity with speed. The speed of a particle at any instant is the magnitude of the velocity at that instant but the average speed will usually bear no relation to the average velocity. The average velocity over a given time interval is the displacement divided by the time whereas the average speed is the total distance travelled divided by the time. If the average velocity is the same as the average speed for all time intervals we say that the particle is moving with uniform (or constant) velocity. Note that this implies a constant direction of motion as well as a constant speed. More generally we have

$$v = \lim_{\delta t \to 0} \left(\frac{\delta s}{\delta t}\right) = \frac{\mathrm{d}s}{\mathrm{d}t}$$

where s represents displacement, v the velocity and t the time. The basic unit is the metre per second (m/s or ms^{-1}).

The ACCELERATION of a particle is defined to be the rate of change of velocity and so is again a vector quantity. There is no special name for the scalar equivalent, it being common practice to still use “acceleration” to describe the rate of change of speed.

$$\text{Clearly } \lim_{\delta t \to 0} \left(\frac{\delta v}{\delta t}\right) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}.$$

The basic unit is the metre per second per second ($\mathrm{m/s}^2$ or ms^{-2})

Motion of a particle along a straight line.

When the motion takes place along a straight line we can neglect the vector nature of displacement, velocity and acceleration except for denoting a reversal of direction by appropriate negative values.

Ex. A particle moves in a straight line with acceleration $-3t\ \mathrm{ms}^{-2}$. The particle is initially passing through some fixed point O with a velocity of $15\ \mathrm{ms}^{-1}$. Find:

(a) the velocity and displacement from O after t seconds, (b) where the particle comes to instantaneous rest, (c) the displacement of the particle in the fourth second, (d) the distance moved during the fourth second.

(a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -3t \Rightarrow v = -\frac{3}{2}t^2 + c$ and $v = 15$ when $t = 0 \Rightarrow c = 15$ so $v = 15 - \frac{3}{2}t^2$ (1)

similarly, $s = \int\left(15 - \frac{3}{2}t^2\right)\mathrm{d}t = 15t - \frac{1}{2}t^3 + c'$ and $s = 0$ when $t = 0$ gives $c' = 0$ so $s = 15t - \frac{1}{2}t^3$ (2)

(b) The particle is at rest when $v = 0$, so from (1) $15 - \frac{3}{2}t^2 = 0 \Rightarrow t^2 = 10 \Rightarrow t = \pm\sqrt{10}$ but only the positive answer is required so $t = \sqrt{10}$ then from (2) $s = 15\sqrt{10} - 5\sqrt{10} = 10\sqrt{10}$

(c) The fourth second is the interval from $t = 3$ to $t = 4$.

$t = 3 \Rightarrow s_1 = 45 - 13.5 = 31.5$ and $t = 4 \Rightarrow s_2 = 60 - 32 = 28$ thus $s_2 - s_1 = -3.5$

(d) $t = 3 \Rightarrow v = 15 - 13.5 = 1.5$ whilst $t = 4 \Rightarrow v = 15 - 24 = -9$

We see therefore that the direction of motion is reversed during the fourth second as we could also have deduced from the fact that the velocity is instantaneously zero when $t = \sqrt{10}$

$t = \sqrt{10} \Rightarrow s = 10\sqrt{10}$ from (b) and so the distance travelled during the fourth second is:

$(s - s_1) + (s - s_2) = (10\sqrt{10} - 31.5) + (10\sqrt{10} - 28) = 20\sqrt{10} - 59.5 = 0.123 + 3.623 = 3.746$

It is good practice to collect the answers together at the end of the question and to give them in complete sentences, thus

(a) The velocity and displacement from O at time t are $\left(15 - \frac{3}{2}t^2\right)\ \mathrm{ms}^{-1}$ and $\left(15t - \frac{1}{2}t^3\right)$ m

(b) The particle comes to instantaneous rest at a distance of $10\sqrt{10} = 31.6$ m from O

(c) The particle undergoes a displacement of –3.5 m during the fourth second.

(d) The particle travels a distance of 3.746 m during the fourth second.

Ex. A particle travelling along a straight line is initially 3 m from a fixed point O of the line. The velocity after t seconds is given by $v = \sin 2t$ ms^{-1}. Find (a) the acceleration when $t = \frac{\pi}{6}$, (b) the greatest displacement of the particle from O during the ensuing motion.

(a) $v = \sin 2t \Rightarrow a = \frac{dv}{dt} = 2\cos 2t$ hence when $t = \frac{\pi}{6},\ a = 2\cos\frac{\pi}{32} = 1$

(b) $\frac{ds}{dt} = \sin 2t \Rightarrow s = -\frac{1}{2}\cos 2t + k$ and $s = 3$ when $t = 0 \Rightarrow k = 3.5$ hence, $s = 3.5 - \frac{1}{2}\cos 2t$

for real values of t, $-1 \le \cos 2t \le 1$ and so the maximum displacement occurs when $\cos 2t = -1$ and $s = 4$

Thus the acceleration after $\frac{\pi}{6}$ secs is 1 ms^{-2} and the greatest displacement from O is 4 m.

A graphical approach

It can often be very helpful to sketch a displacement-time graph and/or a velocity-time graph for the motion of a particle. In both cases it is conventional to plot time on the horizontal axis. The following facts are reasonably obvious:

1. A straight line on the $s - t$ graph $\Rightarrow$ uniform velocity.
2. A straight line on the $v - t$ graph $\Rightarrow$uniform acceleration
3. The gradient of the graph measures the velocity or acceleration
4. The area under a $v - t$ graph measures displacement.

Problems involving experimental results may be solved by drawing accurate graphs

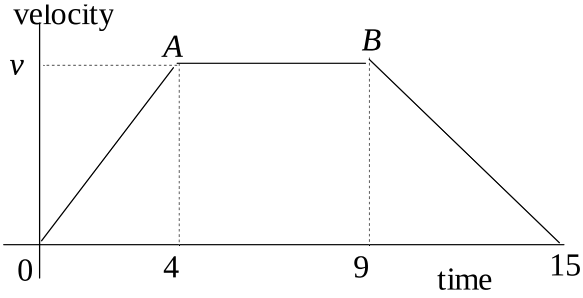
Ex. A body moving in a straight line starts from rest and has an acceleration of 3 ms^{-2} for the first 4 secs Its velocity then remains constant for the next 5 secs and it is brought to rest with uniform retardation in the next 6 secs. Find the total distance travelled.

Velocity-time graph is drawn at the right.

Gradient of $OA = 3$ so $v = 12$

Area under graph $= \frac{1}{2}(5 + 15) \times 12 = 120$

So the body travels a distance of 120 m.



Ex. A car A travels along a straight road with a constant velocity of 30 ms^{-1} for 10 secs, stops for 5 secs t56hen continues at 40 ms^{-1}. A second car B starts from the same point at the same time as A and travels in the same direction at a constant speed of 25 ms^{-1}. Sketch, on the same axes, the $s - t$ graphs for the two cars and use it to find when the cars are level. (Ignore the time taken for A to slow down and speed up again)

The cars are level at points P and Q in the diagram

Car A travels 300 m in the first 10 s$\Rightarrow$ $PN = 300$

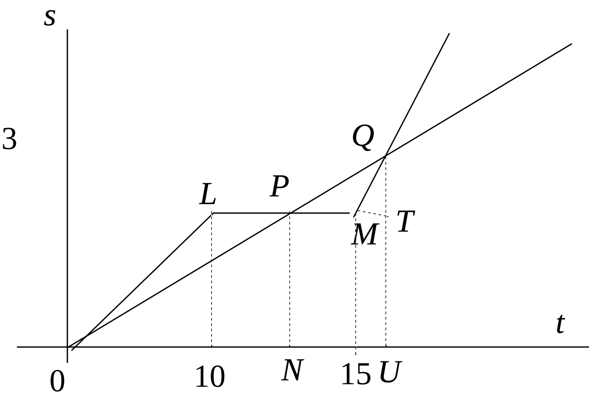
but gradient of $OP = 25 \Rightarrow ON = 12$ hence, $LP = 2 \Rightarrow PM = 3$

Let $MT = x$ then, since gradient of $PQ = 25,\ QT = 25(3 + x)$

but also, gradient $MQ = 40 \Rightarrow QT = 40x$

hence, $25(3 + x) = 40x \Rightarrow 15x = 75 \Rightarrow x = 3$

Thus the cars are level after 0, 12 and 18 secs.



Linear motion with constant acceleration

Consider a particle moving along a straighr line with a uniform acceleration f

Thus $\frac{dv}{dt} = f \Rightarrow v = ft + u$ where U is the initial velocity, then $\frac{ds}{dt} = ft + u \Rightarrow s = \frac{1}{2}ft^2 + ut$ if $s = 0$ at $t = 0$

eliminating u we have $s = \frac{1}{2}ft^2 + (v - ft)t = vt - \frac{1}{2}ft^2$

or, eliminating t, $s = \frac{1}{2}f\left(\frac{v-u}{f}\right)^2 + u\left(\frac{v-u}{f}\right) \Rightarrow 2fs = (v-u)^2 + 2u(v-u) = v^2 - u^2$

or, eliminating f, $s = \frac{1}{2}t^2(\frac{u-v}{t}) + ut = \frac{1}{2}(u+v)t$

Summarising, we have five equations for uniform linear motion.

(1) $v = u + ft$ (2) $s = ut + \frac{1}{2}ft^2$ (3) $s = vt - \frac{1}{2}ft^2$ (4) $s = \frac{1}{2}(u+v)t$ and (5) $v^2 = u^2 + 2fs$

Note that one of the five quantities u, v, f, s and t is omitted from each equation which means that oiven any three of these quantities there will be an equation which will give the value of either of the remaining two.

Ex. A particle moves in a straight line with a uniform acceleration of 8 ms^{-2}. If its initial velocity is 3 ms^{-1},
find (a) its velocity after 4 secs, (b) the distance travelled in 5 secs, (c) the velocity after travelling 7 m. (d) the time taken to travel 20 m.

(a) we have $u = 3$ and $f = 8$ so when $t = 4$, $v = u + ft = 3 + 32 = 35$ ms^{-1}

(b) when $t = 5$, $s = ut + \frac{1}{2}ft^2 = 15 + 100 = 115$

(c) when $s = 7$, $v^2 = u^2 + 2fs = 9 + 16 \times 7 = 121 \Rightarrow v = 11$ (since u and f are positive, v must be)

(d) $s = 20$ so $s = ut + \frac{1}{2}ft^2$ gives $20 = 3t + 4t^2 \Rightarrow 4t^2 + 3t - 20 = 0 \Rightarrow t = \frac{1}{8}(-3 \pm \sqrt{329})$

clearly only the positive solution makes sense so $t = 1.89$

So (a) velocity after 4 secs is 35 ms^{-1}. (b) distance travelled in 5 secs is 115 m. (c) velocity after travelling 7 m is 11 ms^{-1} and (d) time to travel 20 m is 1.89 secs.

Ex. A body travelling in a straight line with a uniform acceleration, has a velocity of 20 ms^{-1} after travelling 1 km and 25 ms^{-1} after t4ravelling 2 km, Find the acceleration and the initial velocity.

With usual notation we have $s_1 = 1000$, $s_2 = 2000$ (note that we muast have s_1 and s_2 in metres)
It may seem at first that we cannot use any of our 5 equations since we do not know the values of any three quantities, but since the initial velocity and acceleration are the same for both distances we can produce a pair of simultaneous equations in u and f,

Using $v^2 = u^2 + 2fs$ we have $20^2 = u^2 + 2000f$ and $25^2 = u^2 + 4000f$

so $25^2 - 20^2 = 2000f \Rightarrow f = \frac{45 \times 5}{2000} = 0.1125$ (note the use of "difference of two squares")

$20^2 = u^2 + 2000f \Rightarrow u^2 = 400 - 225 = 175 \Rightarrow u = 13.2$

So the body has an initial velocity of 13.2 ms^{-1} and an acceleration of 0.1125 ms^{-2}

Vertical motion under gravity.

The acceleration of a freely moving body due to the earth's gravitational pull is found by experiment to be independent of the body itself but inversely proportional to the square of the distance of the body from the centre of the earth. For objects moving near the surface of the earth, this acceleration may be assumed to remain constant, hence vertical motion under gravity is simply a special case of linear motion with constant acceleration.

Consider a body whose distance from the centre of the earth is r metres, then the acceleration f metres/sec^2 due to gravity is given by $f = kr^{-2}$ where k is a universal constant. If the earth is taken to be a perfect sphere of radius R m then the acceleration due to gravity at the surface of the earth is g ms^{-2} where $g = kR^{-2}$

When a body is at a height h m above the earth's surface, assuming $\frac{h}{R}$ *is* small, the acceleration g' is given by $g' = \frac{k}{(R+h)^2} - \frac{k}{R^2}\left(1 + \frac{h}{R}\right)^{-2} \approx \frac{k}{R^2}\left(1 - \frac{2h}{R}\right)$ (using binomial approximation)

Thus the error involved in taking g' to be equal to g is of the order of $2h$ parts in R. then since for motion near the surface of the earth, h is small compared to R, we may safely take the acceleration due to gravity to be constant. The symbol
g is conventionally taken to represent this acceleration and its value, in m, s units varies from about .
9.83 at the poles to 9.78 at the equator

Unless otherwise stated, we use $g = 9.8$.

Consider a body projected vertically upwards with a velocity u ms^{-1}. Assuming that the effect of air resistance is small enough to be neglected the acceleration of the body is $-g$ ms^{-2} The motion is thus linear with constant acceleration and so the equations of the previous section will apply.

So displacement and velocity after t secs are given by $v = u - gt$; $s = ut - \frac{1}{2}gt^2$ and $v^2 = u^2 + 2gs$

<u>Greatest height</u>

The body will be at its greatest height when $v = 0 \Rightarrow t = \frac{u}{g} \Rightarrow$ max height $= u\frac{u}{g} - \frac{1}{2}g\left(\frac{u}{g}\right)^2 = \frac{u^2}{2g}$

<u>Velocity at a given height</u>

When the body is at a height h metres above its point of projection its velocity is given by $v^2 = u^2 - 2gh \Rightarrow v = \pm\sqrt{u^2 - 2gh}$, the sign depending on whether the body is on the way up or down. Note that the speed is the same in each case.

<u>Time to return to point of projection</u>

The body is back at the point of projection when $s = 0$, i.e. when $ut - \frac{1}{2}gt^2 \Rightarrow t = 0$ or $\frac{2u}{g}$

$t = 0$ is obviously when the body is projected so $t = \frac{2u}{g}$ is the time of flight. Note that this is twice the time to reach maximum height.

<u>Ex</u>. A particle is projected vertically upwards from ground level, at 28 ms^{-1}. Find (a) the times when it is 17.5 m above the ground, (b) the greatest height reached, (c) where the speed is 7 ms^{-1}

(a) $s = ut - \frac{1}{2}gt^2 \Rightarrow 17.5 = 28t - 4.9t^2 \Rightarrow 49t^2 - 280t + 175 = 0 \Rightarrow (7t-5)(t-5) = 0 \Rightarrow t = \frac{5}{7}$ or 5

(b) greatest height $= \frac{u^2}{2g} = \frac{28^2}{2\times 9.8} = 40$

(c) $v^2 = u^2 - 2gs \Rightarrow 7^2 = 28^2 - 19.6s \Rightarrow s = \frac{28^2 - 7^2}{19.6} = \frac{35\times 21}{19.6} = 37.5$

Thus (a) the particle is at a height of 17.5 m after $\frac{5}{7}$ s and $5s$, (b) the greatest height attained is 40 m and (c) the particle has a speed of 7 ms^{-1}when at a height of 37.5 m.

<u>Ex</u>. A stone, dropped from the top of a building takes 1 s to fall the last 21 m. Find the height of the building to the nearest tenth of a metre.

Taking downwards as positive, if the height of the building is h m and the total time of fall is t s, then we have $h = 4.9t^2$, since $u = 0$, and $h - 21 = 4.9(t-1)^2 \Rightarrow 4.9(2t-1) = 21 \Rightarrow t = \frac{259}{98} = \frac{37}{14}$

$h = 4.9 \times \frac{37^2}{14^2} = 34.22$. So height of building is 34.2 m.

<u>Newtons Laws of Motion</u>

We now consider the causes of motion and how they are related to the motion they produce.

Consider a book resting on a polished table top. If the book is pushed hard enough it begins to move and continues to do so as long as the push is applied. When we cease to push the book slows down and comes to rest a little further on.

The push applied to the book is an example of a force. When an object is pulled, the pull is another example of a force. Both of these examples involve a physical contact between the object and the agency applying the force but there are other examples of forces which involve no such physical contact, e.g. magnetic or gravitational forces. We define a force as that which causes, or tends to cause motion. The effect of a force is obviously dependent on the direction in which it is applied and it is clear therefore that force is a vector quantity.

When a steadily increasing force is applied to the book, it does not move until the force reaches a certain value. We might think therefore that the book has some built-in resistance to motion which must be overcome before it would move, and this would also explain why the book came to rest after the force was removed. If the book is placed on an unpolished table top however, we find that a considerably larger force is required to cause it to move. This suggests that the resistance to motion is more likely to be due to the surface of the table rather than any inherent property of the book. We therefore postulate that:

(a) the book has no innate resistance to motion

(b) the contact between two surfaces produces a resistance to the motion of one relative to the other.

Thus, when a sufficiently large force is applied, this resistance is overcome and the book begins to move. When the force is removed, the resistance remains and brings the book to rest.

Imagine now a horizontal plane surface extending indefinitely in all directions, its surface so highly polished that no resistance to motion is produced. A body is placed on this surface and set in motion by the application of a force. What will happen when the force is removed?

The postulates we have proposed would suggest that the body will continue to move indefinitely with constant velocity. How do we decide whether to accept or reject such a postulate?

A set of postulates applying to given physical phenomena, are acceptable if they are consistent with the experimental evidence available. So long as the results of experiments continue to be explained and predicted by the postulates, then they are accepted. Inevitably however, there comes a time when new experiments give results which do not agree with those predicted and when this happens, new postulates must be formed to explain not only the results of these new experiments, but also those of all previous experiments which were consistent with the original postulates.

Certain postulates known as Newton's Laws of Motion are accepted providing

(a) the bodies are not of atomic size.

(b) the speed of any body is small compared to the speed of light.

Newton's First Law of Motion

"A body remains in its state of rest or of uniform motion in a straight line, unless compelled to change that state by the action of external forces"

This law defines rest and uniform linear motion to be the two fundamental self perpetuating states of motion.

Before stating the next postulate we require two new concepts.

Mass

The mass of a body is the physical measure of the inertial property of the body, i.e. its resistance to motion. It is often loosely described as the amount of matter contained in the body. Mass is a quantitative concept, a scalar quantity which may be described by a real number of given units. It is our third fundamental concept, the other two being length and time. The basic unit of mass is the kilogram.

Linear Momentum

The linear momentum of a particle is the product of its mass and velocity. The word "linear" is often omitted providing this omission cannot cause confusion. Clearly, momentum is a vector quantity. The momentum of a body is the sum of the momenta of all its constituent particles.

Newton's Second Law of Motion

"The rate of change of linear momentum of a particle with respect to time is proportional to the resultant force acting on the particle"

Thus, if P is the vector sum of all the forces acting on a particle then $\boldsymbol{P} = k\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = k\left(m\boldsymbol{a} + \boldsymbol{v}\frac{\mathrm{d}m}{\mathrm{d}t}\right)$

When m is constant this becomes $\boldsymbol{P} = km\boldsymbol{a}$

The value of$k$ depends on our choice of units.With m in kg, $\boldsymbol{a}$ in ms^{-2}and $\mathbf{P}$ in newtons(N) we have $k = 1$ i.e. we define a force of magnitude 1 N to be that which causes a mass of 1 kg to accelerate at 1 ms^{-2}

Note! Newton's second law is often referred to simply as "the equation of motion"

An alternative unit of force is that which the earth's gravitational attraction exerts on a mass of 1 kg. This is the kilogram - force (kgf) and is known as the gravitational unit of force but suffers from the disadvantage that it varies with the value of g whereas the newton is an "absolute" unit.

Consider a particle of mass 1 kg. If the only force acting upon it is that due to gravity then the acceleration would be g ms$^{-2.}$ 1 kgf $\equiv g$ N

Weight

The gravitational attraction of the earth on a body is known as the "weight" of that body. The weight of a body of mass m kg is therefore m kgf or mg N and always acts vertically downwards. (actually, towards the centre of the earth)

If two objects have masses m_1 kg and m_2 kg then the magnitudes of their weights are W_1 and

W_2 N where $W_1 = m_1 g$ and $W_2 = m_2 g$. Thus providing both objects are at the same place, we have $W_1 : W_2 = m_1 : m_2$ i.e. the ratio of two masses is equal to the ratio of their weights.

Mass may thus be measured by comparing the weight of an object with that of standard masses. We usually assume that:

(a) there is a point of any body which is called the "centre of mass" of that body.

(b) A force applied to a rigid body of constant mass in such a way that its line of action passes through the centre of mass causes an acceleration equal to that of a particle of equal mass under the action of the same force.

Ex. A force of 4 kgf acts on a particle of mass 2 kg, find the acceleration of the particle

Let the acceleration be a ms^{-2}. The force is $4g$ N hence the equation of motion gives

$$4g = 2a \Rightarrow a = 2g = 19.6 \text{ ms}^{-2}$$

Ex. The position vector of a particle of mass 4 kg at time t sec is $2t\mathbf{i}+t^2\mathbf{j}+(3-t^3)\mathbf{k}$ Find the force acting on the particle at time t sec.

Let $\mathbf{r}= 2t\mathbf{i}+t^2\mathbf{j}+(3-t^3)\mathbf{k}$ then $\dot{\mathbf{r}}= 2\mathbf{i}+2t\mathbf{j}-3t^2\mathbf{k}$ and $\ddot{\mathbf{r}}= 2\mathbf{j}-6t\mathbf{k}$ hence force$= 4\ddot{\mathbf{r}}= 8\mathbf{j}-24t\mathbf{k}$

Ex. A particle of mass 3 kg travelling with a velocity of 14 ms^{-1} is brought to rest in 20 m by a constant force. Find the magnitude of this force.

Note that, when the motion is linear we may ignore the vector nature of force and velocity etc.

Let the acceleration be f ms^{-2}.and the force be P N .

Note that f is often used instead of a for accelereation

Using $v^2 = u^2 + 2as$ we have $0 = 196 + 40f \Rightarrow f = -4.9$ hence Force $= ma$ gives $P = 3 \times 4.9 = 14.7$

The force applied is 14.7 N.

Newton's Third Law

" To every action there is an equal and opposite reaction"

Consider two bodies A and B. Newton's third law states that if A exerts any force on B then B exerts an equal and opposite force on A

Ex. A man of mass 70 kg is standing in a lift which is accelerating upwards at 1.4 ms^{-2}. Find the reaction of the lift on the man. If the mass of the lift is 500 kg. Find the tension in the cable.

You should usually begin any solution of a mechanics question with a clear diagram showing all the forces. In a question such as this it is best to have two diagrams, one to show the forces acting on the lift and one to show the forces acting on the man.

Let T N be tension in cable and R N be reaction between man and lift.

Applying Newton's second law to the forces acting On the lift gives $T - (500g + R) = 500 \times 1.4$

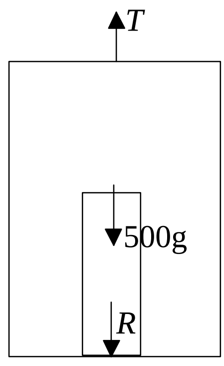
Similarly for the forces acting on the man

$R - 70g = 70 \times 1.4 \Rightarrow R = 70(1.4 + 9.8) = 784$

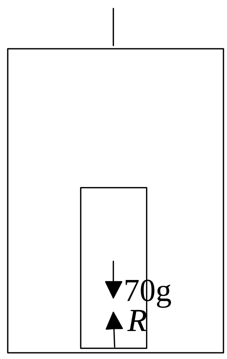
So $T = R + 700 + 500 \times 9.8 = 784 + 5600 = 6384$

So the reaction of the lift on the man is 784 N

and the tension in the lift cable is 6384 N.



Forces on lift



Forces on man

Rigid Bodies

A rigid body is one such that the deformation of shape caused by the forces in the question may be neglected. When such an object moves in such a way that each particle experiences the same displacement in the same time interval (i.e. No rotation is involved), then we may treat the object as a particle of the same mass placed at the centre of mass of the body.

Friction

Consider an object at rest on some horizontal surface. If a gradually increasing force is applied to the object, parallel to the surface, the object at first remains at rest. Newton's laws cause us to conclude that some force is brought into play opposing this applied force and so preventing motion and as the applied force increases so must this opposing force increase. However, eventually the object will begin to move along the surface and so it appears that the applied force will eventually overcome the opposing force. Experiment shows that a constant force slightly less than that required to start the motion, will maintain uniform motion along the surface. In practice we ignore this slight difference. The varying force which opposes motion in this way is called a FRICTION force, and it acts along the surface of contact to directly oppose the tendency to motion.

The following postulates, obtained by experiment, enable predictions to be made.

(1) Before sliding occurs, the friction force is just sufficient to prevent motion.

(2) There is a limiting value to the possible friction force that can exist, and this is proportional to the normal contact force between the surfaces. i.e. $F \le \mu N,$ the constant μ being known as the COEFFICIENT OF FRICTION.

(3) The value of μ depends only on the nature of the surfaces in contact.

(4) When relative motion takes place, the friction force remains constant at its limiting value, or slightly less, and directly opposes the motion.

We may occasionally wish to distinguish between the coefficient of static friction and the coefficient of dynamic friction. The former is the ratio of the friction force to the normal reaction at the instant that sliding starts. The latter is the ratio of the friction force to the normal reaction whilst sliding is taking place. However we very rarely do this and almost always consider them to be the same.

A SMOOTH surface is one for which we take $\mu = 0$ i.e. The friction force may be neglected.

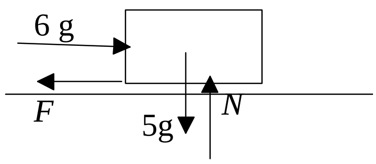
Ex. A particle of mass 5 kg is pushed across a horizontal plane by a horizontal force of 6 kgf. Find the acceleration of the particle if the coefficient of friction between the particle and the plane is 0.25

Taking **i** and **j** to be unit vectors in the direction of motion and vertically upwards respectively, the equation of motion is

$(6g - 0.25N)\mathbf{i} + (N - 5g)\mathbf{j} = 5f\mathbf{i}$

Hence, $6g - 0.25N = 5f$ and $N = 5g \Rightarrow 4.75g = 5f$ so $f = 0.95 \times 9.8 = 9.31$

i.e. acceleration of particle across the plane is 9.31 ms^{-2}



Ex. A particle of mass m kg slides down a line of greatest slope of a plane inclined at an angle θ to the horizontal. The coefficient of friction is μ and the acceleration due to gravity is g ms^{-2}. If t is the time for the particle to attain a speed of v ms^{-1} from rest, find t in terms of v, μ and θ if $\tan\theta > \mu$.

Taking **i** and **j** as indicated, equation of motion is

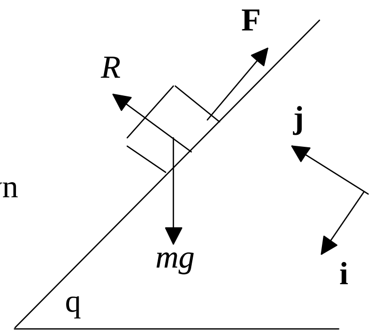
$(mg\sin\theta - F)\text{i} + (R - mg\cos\theta)\mathbf{j} = mf\mathbf{i}$

Hence, $R = mg\theta$ and $mg\sin\theta - F = mf,$ also $F = \mu R$

so $mg\sin\theta - \mu mg\cos\theta = mf$

(Here we can see why the condition $\tan\theta < \mu$ since for movement down the plane we obviousy require that $mg\sin\theta > \mu mg\cos\theta \Rightarrow \tan\theta > \mu$)

Using $v = u + at,\ t = \frac{v-u}{a} = \frac{v}{f} = \frac{v}{(\sin\theta - \mu\cos\theta)g}$



The Tension in a String (or spring)

A "light string (or spring)" is one of negligible mass. When a string is taut, adjacent points exert equal and opposite reactions on each other. The magnitude of this reaction at any point is called the TENSION in the string at that point. A light string necessarily has a constant tension throughout its length. An "inextensible string" is one of constant length"

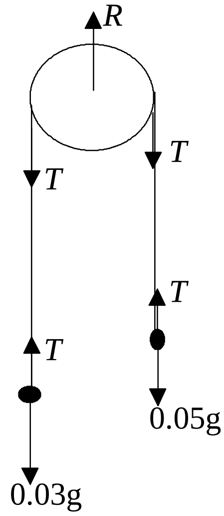
Pulleys "A small smooth pulley" is one of negligible mass and radius with a frictionless axle and so serves only to alter the direction of a string without affecting the tension.

Ex. Two particles of masses 0.05 kg and 0.03 kg are connected by a light inextensible string passing over a small smooth fixed pulley. The portions of the string not in contact with the pulley hang vertically. Find the acceleration of the system, the tension in the string and the reaction of the axle on the pulley.

Forces acting on each mass are its weight and the tension in the string.

Forces acting on the pulley are the tension in the string on either side and the reaction of the axle.

If the acceleration of the system is f ms^{-2} then the 0.05 kg mass will experience a downward acceleration of f ms^{-2} while the 0.03 kg mass experiences an upward acceleration of f ms^{-2}. So applying Newton's second law,



For 0.05 kg mass, $0.05g - T = 0.05f$ and for 0.03 kg mass $T - 0.03g = 0.03f$

So adding together we have $0.02g = 0.08f \Rightarrow f = 0.25g = 2.4525$

hence, $T = 0.05g - 0.05f = 0.0375g = 0.3679$

and for the pulley $R - 2T = 0 \Rightarrow R = 2T = 0.7358$

so acceleration of system is 2.45 ms^{-2}, tension in string is 0.3679 N and the reaction of the axle on the pulley is 0.736 N (all answers to 3.s.f.)

Statics of a Particle

Statics is the branch of Mechanics concerned with systems at rest or in uniform linear motion.

A system in such a state is said to be in EQUILIBRIUM, hence, by Newton's second law, the vector sum of all the forces acting on any particle must be zero.

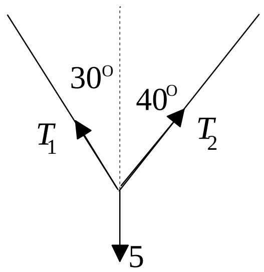
Ex. A particle of mass 5 kg is supported by two strings which are on opposite sides of the vertical at angles of 30^o and 40^o to the vertical. Find the tensions in the strings if the particle is in equilibrium.

Note! When no acceleration is involved there is no necessity to work in absolute units, though you can if you wish.

Before tackling this problem, recall the concept of the "resolved part" of a vector.

i.e. The resolved part of a vector $\mathbf{r}$ in a direction making an angle θ with the vector is $|\mathbf{r}|\cos\theta$.

It follows that if the vector sum of a system of forces is zero, then so also must be the sum of the resolved parts in any given direction



Thus resolving in a direction perpendicular to that of T_2 we have

$T_1\cos 20^o - 5\cos 50^o = 0 \Rightarrow T_1 = \frac{5\cos 50^o}{\cos 20^o} = 3.42$

Similarly, resolving perpendicular to T_1

$T_2\cos 20^o - 5\cos 60^o = 0 \Rightarrow T_2 = \frac{5\cos 60^o}{\cos 20^o} = 2.66$

Thus the tensions are 3.425 and 2.66 kgf respectively

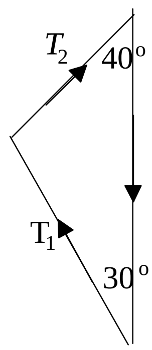
Note that the resolved part of any force in a direction perpendicular to the force is necessarily zero.

A problem such as this can also be solved using the concept of the triangle of forces

If three forces are in equilibrium then you can draw a closed triangle with the sides representing those forces in both magnitude and direction.

From the triangle of forces on the right by the sine rule

We immediately have $T_1 = \frac{5\sin 40^o}{\sin 110^o} = 3.42$ and $T_2 = \frac{5\sin 30^o}{\sin 110^o} = 2.66$



Ex. The forces $(3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}), (2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$ and $(\mathbf{i} - \mathbf{j})$ N act on a particle.

Find the magnitudes of the forces which must be applied in the directions defined by $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$, $-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $-\mathbf{k}$ in order to maintain equilibrium.

Let the required forces be $-a\mathbf{i}-2a\mathbf{j}-2a\mathbf{k}$, $-2b\mathbf{i}-b\mathbf{j}+2b\mathbf{k}$ and $-c\mathbf{k}$ then

$(3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + (\mathbf{i} - \mathbf{j}) + (-a\mathbf{i} - 2a\mathbf{j} - 2a\mathbf{k}) + (-2b\mathbf{i} - b\mathbf{j} + 2b\mathbf{k}) + (-c\mathbf{k}) = 0$

Hence, (1) $3+2+1-a-2b=0$: (2) $4+3-1-2a-b=0$ and (3) $-2+5-2a+2b-c=0$

i.e. (4) $a+2b=6$: (5) $2a+b=6$ and (6) $2a-2b+c=3$

From (4) and (5) $3b=6 \Rightarrow b=2$ then from (4) $a=2$ and finally from (6) $c=3$

The required forces are thus $-2\mathbf{i}-4\mathbf{j}-4\mathbf{k}$, $-4\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and -3k N with magnitudes 6, 6 and 3 Newtons respectively.

Ex. A ring of mass 3 kg is threaded onto a smooth straight rigid wire which is parallel to $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$
The ring is held in equilibrium by a force $(a\mathbf{i}+3\mathbf{j}+6\mathbf{k})$ N. Find a given that the acceleration due to gravity is $-9.8\mathbf{k}$ ms^{-2}

Let the normal reaction of the wire on the ring be $\mathbf{R}$ N

By Newton's second law $\mathbf{R}+3(-9.8\mathbf{k})+(a\mathbf{i}+3\mathbf{j}+6\mathbf{k})=0 \Rightarrow \mathbf{R}=-a\mathbf{i}-3\mathbf{j}+23.4\mathbf{k}$

$\mathbf{R}$ is perpendicular to the wire hence $\mathbf{R}.(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0$ (Scalar product of vectors)

$\Rightarrow (-a\mathbf{i}-3\mathbf{j}+23.4\mathbf{k}).(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0 \Rightarrow -3a-6+23.4=0 \Rightarrow a=\frac{17.4}{3}=5.8$

The Angle of Friction.

Consider a body in equilibrium on a rough surface. Let the normal reaction and friction forces on the body be $R\mathbf{i}$ and $F\mathbf{j}$ N respectively. If $\mathbf{T}$ N is the total reaction of the surface on the body then $\mathbf{T}=R\mathbf{i}+F\text{j}$
The angle θ made by $\mathbf{T}$ with the direction of the normal reaction is given by $\tan\theta=\frac{F}{R}$

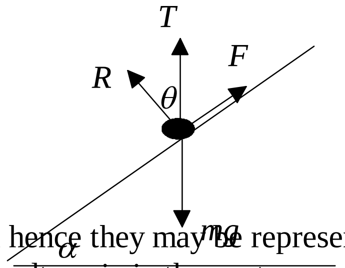
But $\frac{F}{R}\le\mu \Rightarrow \tan\theta\le\mu$

The greatest angle that $\mathbf{T}$ can make with the normal is thus $\arctan\mu$ and is usually denoted by λ

λ is called the ANGLE OF (LIMITING) FRICTION. Thus in any equilibrium situation $\mathbf{T}$ must lie inside, or on the surface of a cone with its axis along the normal to the surface and with semi-vertical angle λ. This is called the CONE of FRICTION.

Ex. A particle of mass m kg is in equilibrium on a plane inclined at an angle α to the horizontal. Find the greatest value of α for which equilibrium is possible if the coefficient of limiting friction is μ

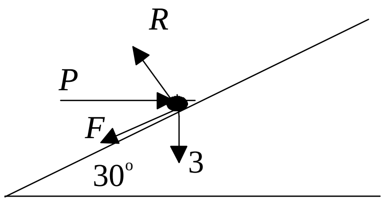
The greatest value of α corresponds to the particle being on the point of sliding down the plane. Combining R and F into a single force T, T must be equal and opposite to mg. Hence, $\theta=\alpha$ (see diagram) but $\theta\le\tan^{-1}\mu \Rightarrow \alpha\le\tan^{-1}\mu$ and the greatest value of α is thus λ.



Note the use here of an important general result.

If three forces acting on a particle are in equilibrium, then each of them must be equal and opposite to the resultant of the other two and hence they may be represented by a TRIANGLE of FORCES. The converse also applies. We use this result again in the next example.

Ex. A particle of mass 3 kg is placed on a plane inclined at 30^o to the horizontal and $\mu=0.5$
A horizontal force of P kgf is applied to the particle so that it is on the point of moving up a line of greatest slope of the plane. Find P

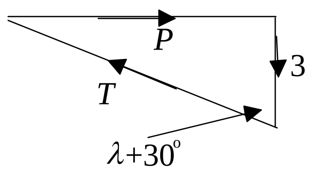


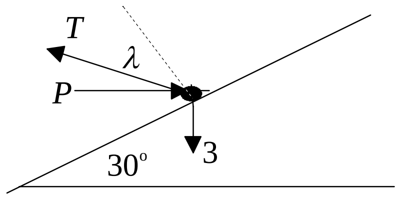
The forces (kgf) are as shown in the diagram
Combining the normal reaction and friction forces gives the second diagram

We have $F=0.5R$ and $\lambda=\tan^{-1}0.5$ so $T=\frac{R\sqrt{5}}{2}$

Since we now only have three forces we may use a triangle of forces.
and it is clear that $P=3\tan(\lambda+30^o)$

$$=\frac{3(\tan\lambda+\tan 30^o)}{1-\tan\lambda\tan 30^o}$$





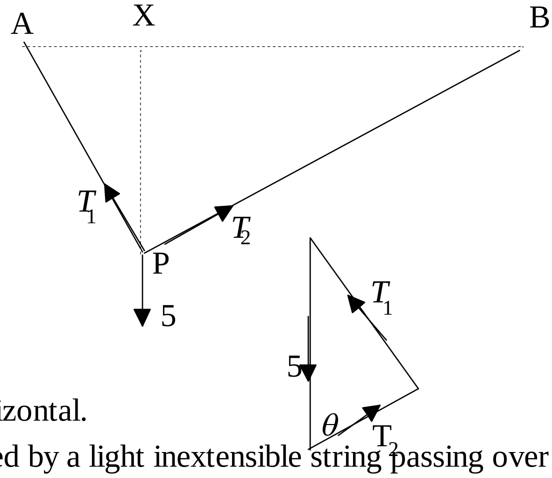
Hooke's Law

Consider an elastic string of unstretched length l metres. This is also called the NATURAL length of the string. Let T N be the tension in the string when it is extended by an amount x metres

It is found by experiment that the relationship between T and x is $T=\frac{\lambda x}{l}$ providing no permanent distortion occurs. λ is a constant called the MODULUS of ELASTICITY of the string (not to be confused with the angle of friction). This result is known as HOOKE'S LAW.

Putting $x=l$ we see that λ is the magnitude of the tension when the length of the string is doubled. Thus λ is measured in the units of force. Hooke's law also applies to the extension and compression of springs within certain limits.

Ex. A particle P of mass 5 kg is suspended by strings AP and BP attached to fixed points A and B at the same horizontal level and 1.3 m apart. In equilibrium AP=0.5 m and BP=1.2 m. If BP is an elastic string of natural length 1 m, find the modulus of elasticity for this string.



Since $AB^2=AP^2+BP^2$ $\quad \angle APB=90^o$

$\Rightarrow X\hat{A}P=X\hat{P}B=\theta$ where $\tan\theta=\frac{12}{5}$

From a triangle of forces

$$T_1=5\cos\theta=5\times\frac{5}{13}=\frac{25}{13}$$

Thus if the modulus of BP is λ then, by Hooke's law

$$\frac{25}{13}=\frac{\lambda\times 0.2}{1}\Rightarrow\lambda=\frac{25}{2.6}=9.62$$

Hence modulus of elasticity of BP is 9.62 kgf

Harder Ex.

A double inclined plane has faces at angles α and β to the horizontal.

A particle of mass m kg rests on the first plane and is connected by a light inextensible string passing over a smooth light pulley at the vertex of the planes to a particle of mass M kg resting on the second plane. The system is in limiting equilibrium with the mass M on the point of sliding downwards. Find the coefficient of friction in terms of m, M, α and β given that it is the same for both planes.

Resolving along the planes

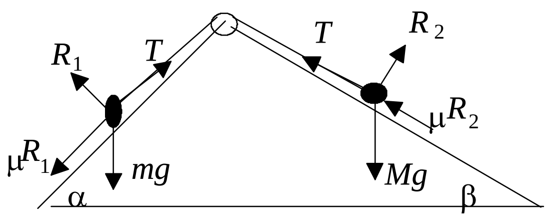
For mass m : $T-\mu R_1-mg\sin\alpha=0$ (1)

For mass M : $T+\mu R_2-Mg\sin\beta=0$ (2)

Resolving perpendicular to planes

For mass m : $R_1-mg\cos\alpha=0$ (3)

For mass M : $R_2-Mg\cos\beta=0$ (4)



From (1) and (3) $T-\mu mg\cos\alpha-mg\sin\alpha=0$

From (2) and (4) $T+\mu Mg\cos\beta-Mg\sin\beta=0$

Hence eliminating T we have $\mu g(M\cos\beta+m\cos\alpha)-g(M\sin\beta-m\sin\alpha)=0$

$$\text{So }\ \mu=\frac{M\sin\beta-m\sin\alpha}{M\cos\beta+m\cos\alpha}$$

Work done by a force

Consider a variable force **F** which moves its point of application along some curve from A to B. Let P and Q be two points on this curve whose position vectors are **r** and **r**+δ**r** respectively where δ**r** is small. Let the force have values **F** and **F**+δ**F** at P and Q.

The work done by the force in the displacement from A to B is

defined to be $\displaystyle\lim_{\delta\mathbf{r}\to\mathbf{0}}\sum_A^B\mathbf{F}.\delta\mathbf{r}=\int_a^b\mathbf{F}.d\mathbf{r}$ (Note the scalar product)

The unit of work is the joule (J)

Now let the force **F** remain constant, then the work done, W is given by

$$W = \lim_{\delta \mathbf{r} \to \mathbf{0}} \sum_A^B \mathbf{F}.\delta\mathbf{r} \quad \lim_{\delta \mathbf{r} \to \mathbf{0}} \mathbf{F}.\sum_A^B \delta\mathbf{r} = \mathbf{F}.\mathbf{AB} = \mathbf{F},\mathbf{s} \text{ Where } \mathbf{s} = \mathbf{AB}$$

Hence, if the angle between **F** and **s** is θ then $W = Fs\cos\theta$

Particular cases.

(1) If the force is constant in magnitude and acts along the path throughout the motion then $W = Fs$ where s is the length of the path.

(2) If the force is constant and acts in the opposite direction to the motion, e.g. friction, then $W = -Fs$ and we say that work is being done against the force.

(3) A force which acts in a direction perpendicular to the motion does no work. In particular, normal reactions and tension in strings with one end fixed can do no work.

Ex. A force of $(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ N moves a particle from the point $(0, 1, 4)$ to the point $(3, 2, 1)$. The units of distance being metres. Find the work done by the force.

Displacement= $(3\mathbf{j} + 2\mathbf{j} + \mathbf{k}) - (\mathbf{j} + 4\mathbf{k}) = 3\mathbf{i}+\mathbf{j}-3\mathbf{k}$ hence $W = (\mathbf{i} - \mathbf{j} + 3\mathbf{k}).(3\mathbf{i}+\mathbf{j}-3\mathbf{k}) = -7$ J

A force is said to be CONSERVATIVE when the work done in a displacement from A to B is independent of the path taken. Any force which is constant in magnitude and direction is necessarily conservative since we then have $W = \mathbf{F}.\mathbf{AB}$

Elastic Strings and Springs

Consider an elastic string of natural length l m and modulus λ N. Let the string be stretched from a length $(l + x1)$ m to $(l + x_2)$ m.

When the string has length $(l + x_1)$ m, the tension is $\frac{\lambda x_1}{l}$, hence, the work done by the tension during a small extra displacement δx along the line of the string is $-\frac{\lambda x_1 \delta x}{l}$ and the total work done is thus

$$\int_{x_1}^{x_2} -\frac{\lambda}{l} x\mathrm{d}x = -\frac{\lambda}{2l}(x_1^2 - x_2^2) = \frac{1}{2}(T_1 + T_2)(x_2 - x_1) \text{ J i.e. average tension times the extension.}$$

Energy

Energy is the measure of the capacity of an object to do work. There are two fundamental types of mechanical energy, KINETIC and POTENTIAL.

Kinetic energy is the capacity to do work by virtue of motion.

Consider a particle moving under the action of a constant force **F** Let the velocity change from v_1 to v_2 during a displacement s, then $\mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{F}.\mathbf{s} = m\mathbf{a}.\mathbf{s} = m\left(\frac{v_2^2 - u^2}{2} - \frac{v_1^2 - u^2}{2}\right) = \frac{1}{2}m(v_2^2 - v_1^2)$

The quantity $\frac{1}{2}mv^2$ is called the “kinetic energy” of the particle and we thus have

Work done = change in kinetic energy

This is true in general providing ALL the forces acting on the particle are taken into account when calculating the work done. This is the WORK-ENERGY principle.

Potential energy is the capacity to do work by virtue of position or shape.

The gravitational potential energy of a particle with respect to some specified horizontaL reference level is the work done by the gravitational force as the particle returns to this reference level.

ie. Potential energy $= mgh$ J where h is the height of the particle above the reference level.

The elastic potential energy stored in a spring, or elastic string, is the work done by the tension as the spring returns to its natural length.

i.e. potential energy = average tension times extension.

Conservation of energy

One of the basic principles of physical science is the PRINCIPLE OF CONSERVATION OF ENERGY. This states that the total amount of energy in the universe is constant. i.e. Energy can be neither created nor destroyed but merely changed from one form to another. A consequence of this law is that in any mechanical system, the total gain in energy of the system is equal to the energy entering the system. In particular, if all the forces are conservative then

Gain of kinetic energy = loss of potential energy.

Ex. 3300 litres of water are raised 10 m through a pipe of radius 10 cm in one minute. Find the work done by the pump in that time ignoring any friction losses etc.

Work done by pump + work done by gravity = gain in kinetic energy

now, rate of flow of water = 3300 litres/min = 0.055 m^3s^{-1}

cross- section area of pipe $=0.01^2 \times \pi = 0.0314$ m^2

hence, speed of water at delivery $= \frac{0.055}{0.0314} = 1.75$ ms^{-1}

thus, gain in k.e. $= 0.5 \times 3300 \times 1.75^2 = 5054$ J

work done by gravity $= -3300 \times 9.8 \times 10 = -323\ 400$ J

hence, work done by pump $= 328454$ J or 3.28×10^5J (3 s.f)

Ex. A particle of mass m kg is attached to a point O on a smooth plane inclined at an angle α to the horizontal by an elastic string of natural length a m and modulus mg N. The particle is projected from O with a velocity $\sqrt{(4ag\sin\alpha + ag)}$ up a line of greatest slope. Find the distance travelled before the particle comes to instantaneous rest.

For the first a m the string is slack. The work done on the particle during this part of the motion is $-mga\sin\alpha$ (the change in p.e.) The initial k.e. is $\frac{1}{2}m(4ag\sin\alpha + ag)$ which is greater than $mag\sin\alpha$, hence the particle does not come to rest before the string becomes taut.

Let the particle come to rest a distance $(a+x)$ m from O.

Since the system is conservative, loss of k.e. = gain of p.e.

i.e. $\frac{1}{2}m(4ag\sin\alpha + ag) = mg(a+x)\sin\alpha + \frac{mgx}{2a}x$ (gain of gravitational and elastic p.e)

$\Rightarrow 4a^2g\sin\alpha + a^2g = 2a^2g\sin\alpha + 2agx\sin\alpha + gx^2$

$\Rightarrow x^2 + 2ax\sin\alpha - a^2(2\sin\alpha + 1) = 0$

$\Rightarrow (x-a)(x + a\{2\sin\alpha + 1\}) = 0 \Rightarrow x = a$ since we know that $x > 0$

Thus the particle comes to rest after travelling $2a$ metres

Power

The rate at which a force is doing work is called the POWER generated by that force.

For a force which does work at a constant rate the power generated is numerically the work done in one second. The unit of power is the watt (W) which is the power generated by a force doing 1 joule of work in 1 second.

Ex. A car is travelling at 36 km/h with its engine generating 20 kW. Find the forward thrust of the engine. (also referred to as the TRACTIVE FORCE of the engine)

The speed of the car $= \frac{36\times1000}{60\times60} = 10$ ms^{-1}

Thus if forward thrust is F N we have $10F = 20000 \Rightarrow F = 2000$

The forward thrust of the engine is therefore 2000 N.

Ex. A car of mass 500 kg generating 21 kW of power is moving along a level road at 42 km/h and has an acceleration of 3 ms^{-2} at this instant. Find the total resistance to motion.

The speed of the car $= \frac{42\times1000}{60\times60} = \frac{35}{3}$ ms^{-1}

If the tractive force is T N then $\frac{35}{3}T = 21000 \Rightarrow T = 1800$

If total resistance to motion is R N then $1800 - R = 500 \times 3 \Rightarrow R = 300$

So the total resistance to motion is 300 N

Ex. A train of mass 200 tonnes climbs an incline of 1 in 200 at a constant speed of 36 km/h.

Find the power being generated by the engine if the resistances to motion, excluding gravity, are 10 N per tonne.

The speed of the train is $\frac{36\times1000}{60\times650} = 10$ ms^{-1} and total resistance to motion $= 200 \times 10 = 2000$ N

The equation of motion for the train gives $T - 2000 - 2\times10^5 \times g \times \frac{1}{200} = 0$

$$\Rightarrow T = 2000 + 9800 = 11\ 800$$

The power generated is thus 118,000 W i.e. 118 kW

Impulse and Momentum

Consider a particle of mass m kg acted upon by a resultant force of $\mathbf{F}(t)$ N during the time interval t_1 to t_2 s. By Newton's second law we have $m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F}(t)$

Integrating with respect to t from t_1 to t_2 we have $m(\mathbf{v}_1 - \mathbf{v}_2) = \int_{t_1}^{t_2} \mathbf{F}(t)\mathrm{d}t$

The expression $\int_{t_1}^{t_2} \mathbf{F}(t)\mathrm{d}t$ is defined to be the IMPULSE (I) of the F(t) during the interval from t_1 to t_2

Hence, the change in momentum of a particle during a given time interval is equal to the impulse of the resultant force on the particle during that interval.

Impulsive forces

When a force of large magnitude acts for a short duration it is often described by its impulse. For example, when two bodies collide we speak of the mutual impulse between them. Forces of this type are called IMPULSIVE FORCES.

Collision of two particles.

Consider two particles of masses m_1 and m_2 which collide with velocities u and u_2. Let the velocities after the collision be v_1 and v_2. Newton's third law tells us that equal and opposite forces act on the particles during the time that they are in contact, hence they must experience equal and opposite impulses and hence undergo equal and opposite changes of momentum.

Thus $m_1(v_1 - u_1) = -m_2(v_2 - u_2) \Rightarrow m_1v_1 + m_2v_2 = m_1u_1 + m_2v_2$

i.e. total momentum after impact = total momentum before impact

This is an example of the LAW of CONSERVATION of MOMENTUM. which states

"In the absence of external forces, the total momentum of a system remains constant"

Ex. A bullet of mass 0.05 kg travelling at 1050 ms^{-1} strikes a block of wood of mass 1 kg which is at rest and embeds itself in this block. If the block is free to move, find the velocity of the block immediately after the impact.

If the velocity of the block is v ms^{-1} then by conservation of momentum we have

$0.05 \times 1050 = 1.05 \times v \Rightarrow v = 50$ so the velocity of the block is 50 ms^{-1}

Ex. A gun of mass 1000 kg which is free to move horizontally, fires a shell of mass 2 kg with a horizontal velocity of 300 ms^{-1}. Find the recoil velocity of the gun.

An explosion exerts forces of equal magnitude in all directions for a short period of time.

The resultant force is thus zero and hence the total momentum of the gun and the shell must remain zero. Thus if the recoil velocity is v ms^{-1} we have

$1000v + 2 \times 300 = 0 \Rightarrow v = 0.6$ so the gun recoils with a velocity of 0.6 ms^{-1}

Ex. A hose delivers 300 litres of water per minute through a 1 cm radius nozzle, The water hits a vertical wall when travelling horizontally at its speed of emission from the hose. If the horizontal velocity is completely destroyed by the impact with the wall, find the force exerted on the wall.

Let the force on the wall be F N. The impulse exerted on the water by the wall is thus $-F$ Ns^{-1} and this must be equal to the change in momentum of the water.

$$\text{Thus } F = \frac{300}{60} \times \frac{0.005}{\pi \times 0.01^2} = 79.6$$

The force exerted on the wall is thus 79.6 N

Projectiles.

Consider a particle projected from a point O with velocity V ms^{-1} at an angle α above horizontal. The particle moves freely under the effect of gravity. Taking $\mathbf{i}$ and $\mathbf{j}$ horizontally and vertically upwards respectively and taking $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ to be the position vector of the particle with respect to O and $\mathbf{v}$ ms^{-1} the velocity after t s then, neglecting air resistance using $\mathbf{r}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$ we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix} \Rightarrow x = t\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

Thus we see that projectile motion is simply a combination of horizontal motion with constant velocity and vertical motion under gravity.

Eliminating t gives $y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$ which is thus the equation of the trajectory.

For given values of x, y and V this can be rearranged into a quadratic equation in $\tan\alpha$ since $\sec^2\alpha = 1+\tan^2\alpha$, hence in general there are two different angles of projection for the particle to pass through a given point. It can be shown that if one of these angles is θ then the other is $90^o - \theta$

Maximum height this clearly occurs when the vertical component of velocity is zero, i.e.$t = \frac{V\sin\alpha}{g}$

Hence, maximum height is $\frac{V^2\sin^2\alpha}{g} - \frac{1}{2}g\frac{V^2\sin^2\alpha}{g^2} = \frac{V^2\sin^2\alpha}{2g}$

Range This is the horizontal distance travelled before returning to ground level and is thus the value of x when $y=0$ i.e. when $Vt\sin\alpha - \frac{1}{2}gt^2 = 0 \Rightarrow t = 0$ or $\frac{2V\sin\alpha}{g}$ so we see that the time of flight is $\frac{2V\sin\alpha}{g}$ (i.e. twice the time to each maximum height)

Hence range $= Vt\cos\alpha = \frac{2V^2\sin\alpha\cos\alpha}{gt}$ or $\frac{V^2\sin 2\alpha}{g}$

Ex. A particle is projected from a point O with a velocity of 15 ms^{-1} at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ to the horizontal. Find the velocity of the particle and its position vector after 2 seconds. Find the speed of the particle when it is $2\frac{6}{7}$ m above the horizontal plane through O.

Since $\tan\alpha = \frac{4}{3}$ we have $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$

$\mathbf{v}=\begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$ so after 2 s, $\mathbf{v}=\begin{pmatrix} 15\times\frac{3}{5} \\ 15\times\frac{4}{5} - 19.6 \end{pmatrix} = \begin{pmatrix} 9 \\ -7.6 \end{pmatrix}$ and

$\mathbf{r}=\begin{pmatrix} 15\times\frac{3}{5}\times 2 \\ 15\times\frac{4}{5}\times 2 - 19.6 \end{pmatrix} = \begin{pmatrix} 18 \\ 4.4 \end{pmatrix}$ thus after 2 seconds $\mathbf{v}= (9\mathbf{i} - 7.6\mathbf{j})$ ms^{-1} and $\mathbf{r}= (18\mathbf{i}+4.4\mathbf{j})$ m

Using $v^2 - u^2 = 2as$ we have $v^2 = V^2\sin^2\alpha - 19.6\times\frac{20}{7} = 15^2\times\left(\frac{4}{5}\right)^2 - 2.8\times 20 = 144 - 56 = 88$

Thus vertical velocity is now $\sqrt{88}$ ms^{-1}, horizontal velocity is still 9 ms^{-1} so

speed $= \sqrt{9^2+88} = \sqrt{169} = 13$ ms^{-1}

Ex. A particle is projected with a velocity of $(V\cos\alpha)\mathbf{i}+(V\sin\alpha)\mathbf{j}$ ms^{-1}. Find the position vector of the particle when its velocity is at right angles to its initial velocity.

$\mathbf{v}=\begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}$ and is at right angles to $\begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix}$ when $\begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}.\begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix} = 0$

i.e. $V^2\cos^2\alpha + V^2\sin^2\alpha - Vgt\sin\alpha = 0 \Rightarrow V^2 - Vgt\sin\alpha = 0 \Rightarrow t = \frac{V}{g\sin\alpha}$

So position of particle is $\begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix} = \begin{pmatrix} \frac{V^2}{g}\cot\alpha \\ \frac{V^2}{g} - \frac{V^2}{2g\sin^2\alpha} \end{pmatrix}$

i.e. position vector of particle is $\left(\frac{V^2}{g}\cot\alpha\right)\mathbf{i}+\left(\frac{V^2}{g}(1-\operatorname{cosec}^2\alpha)\right)\mathbf{j}$

Ex. A man throws a ball with speed u at an angle α to the horizonatl Prove that (a) its maximum height is $\frac{u^2\sin^2\alpha}{2g}$ (b) its greatest horizontal range is four times its maximum height during such a trajectory.

(a) with usual notation, $v_y = 0 \Rightarrow u\sin\alpha - gt = 0 \Rightarrow t = \frac{u\sin\alpha}{g}$

hence maximum height is $Vt\sin\alpha - \frac{1}{2}gt^2 = \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g} = \frac{u^2\sin^2\alpha}{2g}$

(b $y = 0 \Rightarrow ut\sin\alpha - \frac{1}{2}gt^2 = 0 \Rightarrow$ time of flight is $\frac{2u\sin\alpha}{g}$

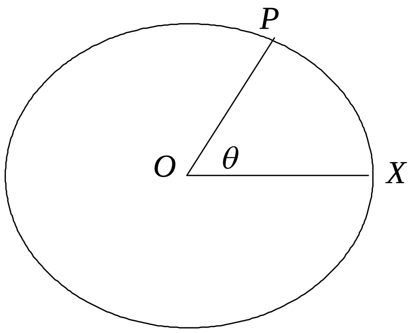
the range is thus $ut\cos\alpha = \frac{2u^2\sin\alpha\cos\alpha}{gt}$ or $\frac{u^2\sin 2\alpha}{g}$ which is clearly a maximum when $\alpha = 45^o$

hence, maximum range is $\frac{u^2}{g}$ and maximum height is $\frac{u^2}{4g}$ and so the maximum range is four times the greatest height.

Uniform circular motion.

Consider a particle moving in a circle centre O, radius a m with constant angular velocity ω rad/s

Let the initial position of the particle be at X and the position after t sec be P. The position vector of P is $\mathbf{r}$ and angle $X\hat{O}P = \theta$



Taking vectors i,j along and perpendicular to OX,

since $\omega = \frac{d\theta}{dt}$ we have $\theta = \omega t$

So $\mathbf{r} = \begin{pmatrix} a\cos\omega t \\ a\sin\omega t \end{pmatrix} \Rightarrow \dot{\mathbf{r}} = \begin{pmatrix} -a\omega\sin\omega t \\ a\omega\cos\omega t \end{pmatrix}$ & $\ddot{\mathbf{r}} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -a\omega^2\sin\omega t \end{pmatrix}$

thus the acceleration of the particle is $-\omega^2\mathbf{r}$

The magnitude of this acceleration is $\omega^2 a$ and its direction is radially towards O.

The speed is $|\dot{\mathbf{r}}| = a\omega$ and the direction of motion is $\tan^{-1}\left(\frac{-\cos\omega t}{\sin\omega t}\right)$ i.e. tangentially to the circle.

If the speed along the path is then v ms^{-1} then $v = \omega a$ and hence the acceleration may also be written as $\frac{v^2}{a}$ towards the centre of the circle.

By Newton's second law it follows that a force of $m\omega^2 a$ or $\frac{mv^2}{a}$ must be applied radially towards the centre of the circle in order to produce uniform circular motion.

Ex. A particle of mass 4 kg is attached by a light inelastic string of length 2 m, to a fixed point O on a smooth table top. The particle rotates about O with the string taut, at a constant speed of 5 ms^{-1}. Find the tension in the string

It vis the tension in the string, acting radially towards O that causes the circular motion so we must have, tension $T = \frac{mv^2}{r} = \frac{4\times 5^2}{2} = 50$. so the tension is 50 N.

Ex. A particle of mass 4 kg is on a smooth horizontal table top. It is attached to a light inelastic string which passes along the table to, through a small hole and has a particle of mass 2 kg hanging freely on the other end. The 4 kg particle rotates about the hole with an angular velocity of 3.5 rad/s and the 2 kg particle remains at rest, Find the distance of the 4 kg mass from the hole.

Let the 4 kg mass be a distance x m from the hole

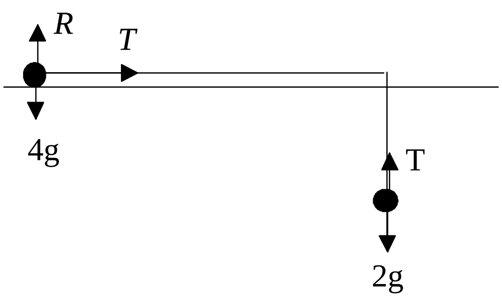
Considering the 2 kg particle we must have $T = 2g$

and for the 4 kg particle, $T = m\omega^2 r = 4\times 3.5^2 x$

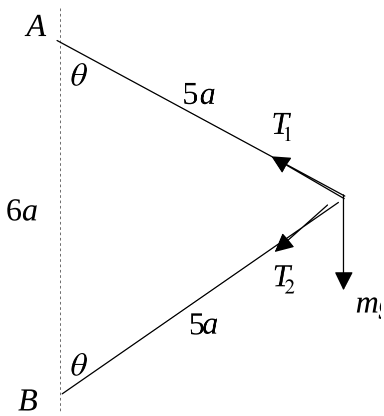
but the tension is constant along the length of the string

so we must have $4\times 3.5^2 x = 2g \Rightarrow x = \frac{19.6}{49} = 0.4$

thus the 4 kg particle is 0.4 m from the hole.



Ex. A and B are points on a fixed vertical pole at a distance of $6a$ m apart. A particle of mass m kg is attached to A and B by light inextensible strings Of length $5a$ nm. Find the tensions in the strings if the particle moves in a horizontal circle with an angular velocity of $\sqrt{\frac{4g}{a}}$ rad/s.



Resolving vertically, $T_1\cos\theta = T_2\cos\theta + mg$

Resolving horizontally $T_1\sin\theta + +T_2\sin\theta = m\omega^2 r$

so since $\sin\theta = 0.8$ and $\cos\theta = 0.6$ we have

$0.8T_1 + 0.8T_2 = \frac{4g}{a}\times 4a\times m = 16mg \Rightarrow T_1 + T_2 = 20mg$

and $0.6T_1 - 0.6T_2 = mg \Rightarrow T_1 - T_2 = \frac{5}{3}mg$

hence, $2T_1 = 21\frac{2}{3}mg$ and $2T_2 = 18\frac{1}{3}mg \Rightarrow T_1 = 10\frac{5}{6}mg$ and $T_2 = 9\frac{1}{6}mg$

hence, tensions are $\frac{65}{6}mg$ N in upper string and $\frac{55}{6}mg$ N in lower string

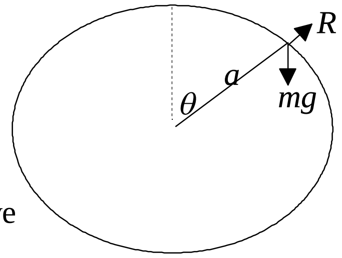
Note! We have assumed that both strings are taut but, if that assumption had been wrong then T_2 would have worked out to be negative which is impossible.

Motion in a vertical circle

Examples of motion in a vertical circle are: a ring sliding on a vertical circular wire: a particle sliding down the outside of a sphere or sliding inside a hollow sphere: the motion of the bob of a simple pendulum.

In each case the weight of the particle will, in general, have a component along the tangent giving the particle a tangential acceleration as well as the radial acceleration. So we can apply Newton's second law along the tangent as well as radially. However it is usually more convenient to make use of the Principle of Work.

Ex. A small ring of mass m kg is threaded on a smooth wire bent into the form of a circle of radius a metres , lying in a vertical plane. the ring is given a speed of u ms^{-1} when it is at the lowest point of the circle. find the speed when it is at a position such that the radius to that point makes an angle θ with the upward vertical. hence find the value of u that will enable the ring to just reach the highest point of the circle.



Since the wire is smooth, the system is conservative, thus

loss of k.e = gain of p.e

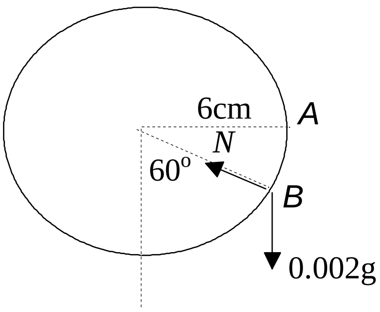
i.e. $\frac{1}{2}m(u^2 - v^2) = mga(1+\cos\theta) \Rightarrow v^2 = u^2 - 2ga(1+\cos\theta)$

and the speed is thus $\sqrt{(u^2 - 2ga(1+\cos\theta))}$ ms^{-1}

If the ring just reaches the highest point of the circle then we must have

$v = 0$ when $\theta = 0 \Rightarrow u^2 = 4ga$ i.e. value of u required is $2\sqrt{ga}$ ms^{-1}

Ex. A small bead of mass 2 g is threaded on a smooth circular wire fixed with its plane vertical. The bead is released from rest in a position where the radius to the bead is horizontal. The radius of the wire is 6 cm. Find the reaction between the bead and the wire when the radius makes an angle of 60^o with the downward vertical.



Let the particle be moving with speed v at the instant when the radius makes an angle of 60^o to the vertical. (see diagram)

Since it is moving in a circle the component of acceleration towards the centre will be $\frac{v^2}{0.06}$ ms^{-2}

So resolving radially towards the centre $N - 0.002g\cos 60^o = 0.002 \times \frac{v^2}{0.06}$

$\Rightarrow N = 0.002\left(\frac{v^2}{0.06} + \frac{1}{2}\right)$ (1)

Applying the Principle of work between positions A and B

Final K.E.-Initial K.E, = Work done I.e. $\frac{1}{2}(0.002)v^2 - 0 = 0.002g(0.06\cos 60^o)$

$\Rightarrow v^2 = 2g(0.06)\cos 60^o$ so substituting in (1) we have

$N - 0.002g\cos 60^o = \frac{0.002}{0.06} \times 2g(0.06\cos 60^o) \Rightarrow N = 0.001g + \frac{g}{15} \times 0.03 = 0.003g$ N

Linear Motion under varying forces.

When the forces causing a body to move in a straight line are not constant, we must NOT use equations such as $v = u + at$ etc. The equation of motion still holds however and may be written in various ways (m is assumed to remain constant)

$\mathbf{F}(t) = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$, $\mathbf{F}(t) = m\frac{\mathrm{d}^2\mathbf{s}}{\mathrm{d}t^2}$, $\mathbf{F}(v) = m\frac{\mathrm{d}v}{\mathrm{d}t}$, $\mathbf{F}(s) = mv\frac{\mathrm{d}v}{\mathrm{d}s}$ etc, depending on what is varying and what we require to find.

Also the principle of energy still holds so always consider the possibility of using this.

Force as a function of time.

Ex. A man pushing a car of mass 600 kg starting from rest exerts a force which is initially 250 N but which decreases steadily with time until, after 20 seconds, it just counteracts the constant resistance of 50 N. How fast is the car then moving and how far has the man pushed it?

Force exerted by man = $(250 - 10t)$ N hence $m\frac{\mathrm{d}v}{\mathrm{d}t} = (250 - 10t) - 50$

$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{3} - \frac{t}{60} \Rightarrow v = \frac{1}{3}t - \frac{1}{120}t^2$ since $v = 0$ when $t = 0$, and so $s = \frac{1}{6}t^2 - \frac{1}{360}t^3$ ($s = 0$ when $t = 0$)

Thus when $t = 20$, $v = \frac{20}{3} - \frac{20}{6} = \frac{10}{3}$ and $s = \frac{200}{3} - \frac{200}{9} = \frac{400}{9}$

So the speed is $3\frac{1}{3}$ ms^{-1} and he has pushed it 44.4 m.

Force as a function of velocity

Ex. A particle of mass m kg falls from rest under gravity against a resistance that is proportional to the velocity. Investigate the motion

We have $m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - mkv \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = g - kv \Rightarrow \int \frac{\mathrm{d}v}{g-kv} = \int \mathrm{d}t$

Hence, $-\frac{1}{k}\ln(g-kv) = t + c$ and $v = 0$ when $t = 0 \Rightarrow c = -\frac{1}{k}\ln g$ so $\frac{1}{k}\ln g = \frac{1}{k}\ln g - t$

$\Rightarrow \ln\left(\frac{g-kv}{g}\right) = -kt \Rightarrow g - kv = ge^{-kt}$ or $v = \frac{g}{k}(1 - \mathrm{e}^{-kt})$

Thus as $t \to \infty$, $v \to \frac{g}{k}$ which is thus the terminal or limiting velocity.

also, $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{g}{k}(1 - \mathrm{e}^{-kt}) \Rightarrow s = \frac{g}{k}\left(t + \frac{1}{k}\mathrm{e}^{-kt}\right) + c'$ and $s = 0$ when $t = 0 \Rightarrow c' = -\frac{g}{k^2}$

so $s = \frac{g}{k^2}(kt + \mathrm{e}^{-kt} - 1)$

Ex. The engine of a car of mass 500 kg is working at a rate of 50 kW. Find the time taken to accelerate from 36 km/h to 72 kmh if the resistance to motion is $20v$ N when the velocity is v ms^{-1}

The tractive force at v ms^{-1} is $50\,000v^{-1}$ N so the equation of motion is

$500\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{50\,000}{v} - 20v \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{2500-v^2}{25v}$

Let T sec be the time taken to accelerate from 36 kmh^{-1}(10 ms^{-1}) to 72 kmh^{-1}(20 ms^{-1}) then

$25\int_{10}^{20} \frac{v\mathrm{d}v}{2500-v^2} = \int_0^T \mathrm{d}t \Rightarrow -\frac{25}{2}[\ln(2500 - v^2)]_{10}^{20} = T \Rightarrow T = -\frac{25}{2}(\ln 2100 - \ln 2400) = 1.67$

The car takes 1.67 seconds to accelerate from 36 km/h to 72 km/h.

Force as a function of distance

Ex. A particle moving along a straight line is subjected to a force which is proportional to the distance of the particle from a fixed point O of the line and is directed towards o. If the particle is initially at rest at a distance a m from O, investigate the motion.

let the particle be at a distance x from O at time t

then $mv\frac{\mathrm{d}v}{\mathrm{d}x} = -mkx\ (k > 0) \Rightarrow \int v\mathrm{d}v = -k\int x\mathrm{d}x + A \Rightarrow \frac{1}{2}v^2 = A - \frac{1}{2}kx^2$

i.e. $v^2 = B - kx^2$ and $v = 0$ when $x = a \Rightarrow B = ka^2$ so $v^2 = k(a^2 - x^2)$ or $v = \pm\sqrt{k(a^2 - x^2)}$

for reasons which will become clear later we usually write the constant k as ω^2 so we have

$v = \pm\omega\sqrt{a^2 - x^2}$

thus $\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\omega\sqrt{a^2 - x^2} \Rightarrow \pm\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \omega\mathrm{d}t \Rightarrow \sin^{-1}(\frac{x}{a}) = \omega t + C$ or $\cos^{-1}(\frac{x}{a}) = \omega t + C$

by the given initial conditions, $C = \frac{\pi}{2}$ in the first case and 0 in the second case. hence the solutions are identical and we may simply write $x = a\cos\omega t$

This is an example of simple harmonic motion (S.H.M.) which is the subject of the next item.

Oscillations

Ex.1 If a spring of modulus λ hangs in equilibrium from a fixed point with a mass m attached to its other end, find the equation of motion that describes the subsequent motion when it is pulled down a further distance a and released.

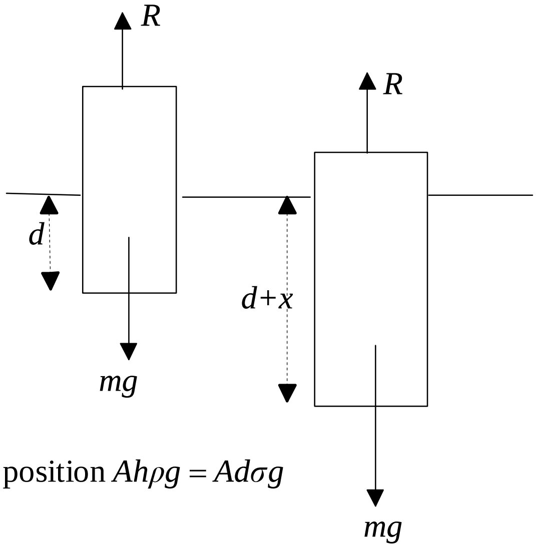
Consider when the mass is a distance x below the equilibrium position. By Hooke's Law, when in equilibrium., the tension is given by $T = \frac{\lambda d}{l}$ where d is the extension and l the natural length.

when the exrtension is $d + x$, $T = \frac{\lambda(d+x)}{l}$ and so, the general equation of motion is

$mg - \frac{\lambda}{l}(d + x) = m\ddot{x} \Rightarrow m\ddot{x} = -\frac{\lambda x}{l}$ since $mg = \frac{\lambda d}{l}$ and so $\ddot{x} = -\frac{\lambda}{ml}x$

Hence the acceleration of the mass is proportional to its displacement from the equilibrium position and is in the opposite direction. The tension in the spring acts as a restoring force, attempting to re-establish equilibrium. So the mass oscillates up and down about the equilibrium position.

Ex.2. Investigate the equation of motion of a cylindrical canister buoy, bobbing up and down in a calm sea.



Let equilibrium position be as shown in the left hand diagram and consider when in position shown on the right Suppose the buoy has a constant cross-section area. A m^2 And height h m.

If the density of the buoy is ρ and that of the sea σ then $mg = Ah\rho g$ and the upthrust R is the weight of water displaced i,e, $R = Ad\sigma g$ in equilibrium position and $A(d+x)\sigma g$ when in position show on the right.

So in this position the equation of motion is

$Ah\rho g - A(d+x)\sigma g = Ah\rho\ddot{x} \Rightarrow -Ax\rho g = Ah\sigma r\ddot{x}$ since in eqilibrium position $Ah\rho g = Ad\sigma g$

Hence, $\ddot{x} = -\frac{\sigma g}{\rho h}x = -\frac{g}{d}x$ since $h\rho = d\sigma$

So again the acceleration of the buoy is proportional to the displacement from the equilibrium position and in the opposite direction to that displacement.

Ex 3 The simple pendulum. A mass m is suspended from a fixed point by a light inextensible string of length l and swings to and fro in a vertical plane. Investigate the motion.

When the string makes an angle θ with the downward vertical the acceleration of the mass has two components. $l\ddot{\theta}$ perpendicular to the string and $l\dot{\theta}^2$ along the string towards the point of suspension.

Hence, equation of motion perpendicular to the string is $-mg\sin\theta = ml\ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$

If the string is swinging through a small angle then $\sin\theta \approx \theta$ and so $\ddot{\theta} \approx -\frac{g}{l}\theta$

In all three of these examples the final equation of motion has the form $\ddot{x} = -kx$ $(k > 0)$

Any motion which can be modelled by such an equation is called SIMPLE HARMONIC MOTION (SHM)

As $k > 0$ we can write $k = \omega^2$ and have $\ddot{x} = -\omega^2 x$ which we will see is a more cconvenient form.

Reweriting this equation as $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$, we know from calculus that $x = A\sin\omega t + B\cos\omega t$ is a general solution, or alternatively $x = C\sin(\omega t + \alpha)$

Clearly, C is the maximum value of x which we usually denote by a and call the AMPLITUDE of the motion. Thus $x = a\sin(\omega t + \alpha)$ hence, $\dot{x} = a\omega\cos(\omega t + \alpha)$ and so $\dot{x}^2 = \omega^2(a^2 - x^2)$

The value of α is determined by the position of the oscillating mass at time $t = 0$

In particular, if $x = 0$ when $t = 0$ then $\alpha = 0$ and $x = a\sin\omega t$

If $x = a$ when $t = 0$ then $\alpha = \frac{\pi}{2}$ and $x = a\cos\omega t$

The period of the motion, i.e. the time to return to its original position is clearly given by $\omega t = 2\pi$ so period $= \frac{2\pi}{\omega}$

so in our three examples we have (i) $\omega^2 = \frac{\lambda}{ml} \Rightarrow t = 2\pi\sqrt{\frac{ml}{\lambda}} = 2\pi\sqrt{\frac{d}{g}}$

(2) $\omega^2 = \frac{\sigma g}{\rho h} \Rightarrow t = 2\pi\sqrt{\frac{\rho h}{\sigma g}} = 2\pi\sqrt{\frac{d}{g}}$ and (3) $\omega^2 = \frac{g}{l} \Rightarrow t = 2\pi\sqrt{\frac{l}{g}}$

Ex. A particle is describing SHM in a straight line. When its distance from O, the centre of its path is 3 m, its velocity is 16 ms^{-1} towards O and its acceleration is 48 ms^{-2} towards O. Find (a) the period of the motion, (b) the amplitude of the motion (c) the time taken for the particle to reach O (d) the velocity of the particle as it passes through O.

(a) With the usual Notation. $\ddot{x} = -\omega^2 x$ and $\ddot{x} = -48$ when $x = 3 \Rightarrow \omega^2 = \frac{-48}{-3} = 16 \Rightarrow \omega = 4$

Hence the period is $\frac{2\pi}{\omega} = \frac{\pi}{2} = 1.57$ secs

(b) $\dot{x}^2 = \omega^2(a^2 - x^2) = 16(a^2 - x^2)$ and $\dot{x} = -16$ when $x = 3 \Rightarrow 256 = 16(a^2 - 9) \Rightarrow a^2 = 25 \Rightarrow a = 5$

Thus the amplitude is 5 m.

(c) Taking $t = 0$ when $x = 0$ we have $x = a\sin\omega t = 5\sin 4t$ hence $x = 3 \Rightarrow 3 = 5\sin 4t$

$\Rightarrow t = \frac{1}{4}\sin^{-1}(0.6) \approx 0.16$ and so the particle takes 0.16 seconds to go from O to the position described and hence, by the symmetry of the motion, it will take the same time to go from this position to O

(d) Finally, $\dot{x}^2 = 16(25 - x^2)$ and $x = 0 \Rightarrow \dot{x}^2 = 16 \times 25 \Rightarrow \dot{x} = 20$

so the particle passes through O at 20 ms^{-1}

Impulse and Momentum again

A DIRECT COLLISION of two bodies occurs when the bodies have velocities along the common normal to their surfaces at the moment of collision. An OBLIQUE COLLISION occurs when one or both of the bodies has a velocity which is not along the common normal.

When two smooth bodies collide, resistance will be offered to the relative motion along the common normal but the components of the velocities perpendicular ton the normal, i.e. along the common tangent plane, will not be changed. This resistance causes the bodies to undergo compression since the normal component of the relative velocity cannot be reduced to zero instantaneously. If this compression is permanent then the bodies will have the same velocity along the normal after the impact. Such bodies are said to be INELASTIC.

More usually, the bodies will expand back again after the maximum compression has been attained though this expansion will not necessarily restore the bodies to their original states. The component of the relative velocity along the normal will therefore be changed from zero and will have a final value in the opposite direction to that before the collision. If the forces of expansion are equal to those of compression then the component of the relative velocity along the normal after the impact will be equal and opposite to that before the impact. Such collisions are said to be PERFECTLY ELASTIC and the bodies involved are PERFECTLY ELASTIC BODIES. Such bodies are mathematical concepts and do not exist in the physical world.

In collisions in the real world the forces of expansion are smaller than those of compression because the latter are helped by internal friction forces whilst the former are opposed by such friction forces. The normal component of relative velocity will thus be smaller in magnitude after a collision. In situations such as this NEWTON’S LAW OF RESTITUTION enables us to predict this change in relative velocity. This law, like most physical laws, is deduced by experiment.

Newton’s law of restitution

When two bodies A and B collide, the ratio of the components of the velocity of A relative to B along the common normal after and before impact is equal to a negative constant. This constant is independent of the velocities of A and B and has a magnitude that is less than unity. The absolute value of this constant is known as the COEFFICIENT OF RESTITUTION for the two bodies and is usually denoted by the letter e.. Thus, if two bodies collide directly with velocities u_1 and u_2 ms^{-1} and rebound with velocities v_1 and v_2 ms^{-1} .with all velocities measured in the same direction then

$\frac{v_1 - v_2}{u_1 - u_2} = -e \Rightarrow v_1 - v_{21} = -e(u_1 - u_2)$

For inelastic bodies $v_1 = v_2 \Rightarrow e = 0$

For perfectly elastic bodies $v_1 - v_2 = -(u_1 - u_2) \Rightarrow e = 1$.

If the bodies collide with velocities $\mathbf{u}_1, \mathbf{u}_2$ and rebound with velocities $\mathbf{v}_1, \mathbf{v}_2$ and $\hat{\mathbf{n}}$ is a unit vector along the common normal then $(\mathbf{v}_1 - \mathbf{v}_2).\hat{\mathbf{n}} = -e(\mathbf{u}_1 - \mathbf{u}_2).\hat{\mathbf{n}}$

Direct impact

Ex. A smooth sphere of mass 3 kg is moving with a velocity of 6 ms^{-1} when it collides directly with a second smooth sphere of mass 4 kg moving with a velocity of –10 ms^{-1}. Find the velocities after the impact if the coefficient of restitution is 0.25.

Let the velocities of the 3 and 4 kg spheres after the impact be v_1 and v_2 respectively.

By conservation of momentum $3v_1 + 4v_2 = 3 \times 6 + 4 \times (-10) = -22$

By law of restitution $v_1 - v_2 = -0.25(6 - (-10)) = -4$

Hence, $7v_1 = -38 \Rightarrow v_1 = -\frac{38}{7}$ and $v_2 = -\frac{10}{7}$

The velocities of the 3 and 4 kg masses are $-\frac{38}{7}$ and $-\frac{10}{7}$ ms^{-1} respectively

Oblique impact of smooth spheres.

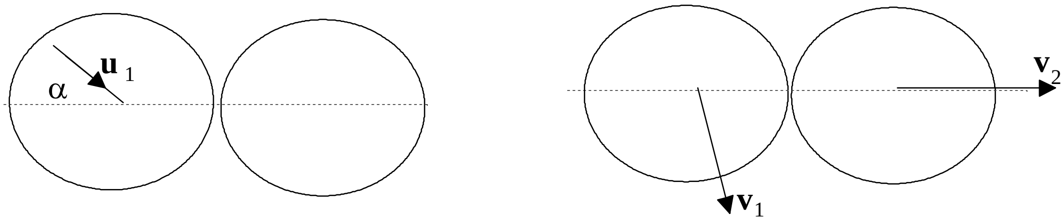
The results governing the oblique impact of two smooth bodies are:

(1) The law of conservation of momentum

(2) Newton's law of restitution

(3) The fact that the resolved parts of the velocities perpendicular to the common normal are unchanged by the impact.

Ex. A smooth sphere of mass m kg collides with an identical sphere which is at rest. The velocity of the first sphere makes an angle α with the line of centres at the moment of impact and is of magnitude u ms^{-1} the coefficient of restitution is e. Find the velocities of the spheres after the impact.



Before impact After impact

Let **i**,**j** be unit vectors along and perpendicular to the line of centres.

Then $\mathrm{u}_1 = \begin{pmatrix} u\cos\alpha, \\ u\sin\alpha \end{pmatrix}$, $\mathbf{u}_2 = 0$, $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ u\sin\alpha \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$

Conservation of momentum along the line of centres gives $mx_1 + mx_2 = mu\cos\alpha \Rightarrow x_1 + x_2 = u\cos\alpha$

Newton's law of restitution gives $x_1 - x_2 = -e(u\cos\alpha - 0)$

Hence, $x_1 = \frac{1}{2}(1-e)u\cos\alpha$ and $x_2 = \frac{1}{2}(1+e)u\cos\alpha$

The velocities of the spheres after impact are $\begin{pmatrix} \frac{1}{2}(1-e)u\cos\alpha \\ u\sin\alpha \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2}(1+e)u\cos\alpha \\ 0 \end{pmatrix}$

Collision with a smooth fixed plane

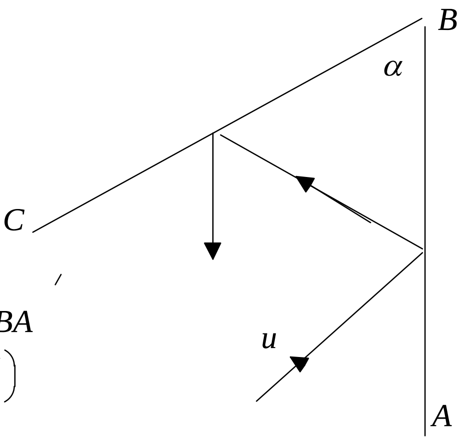
Consider a body moving with a velocity of $u\mathbf{i}+v\mathbf{j}$) ms^{-1}, which collides with a smooth fixed plane such that **i** is a unit vector perpendicular to the plane. Let the velocity after the impact be $(x\mathbf{i}+y\mathbf{j})$ ms^{-1}

Since the plane is smooth we must have $y = v$

Newton's law of restitution gives $x - 0 = -e(u-0) \Rightarrow x = -eu$

Thus the velocity after impact is $(-eu\mathbf{i} + v\mathbf{j})$ ms^{-1}

Ex. Two smooth vertical walls meet a smooth horizontal plane in the lines AB and BC where angle ABC = α. A particle is projected towards AB, parallel to CB, with a velocity of u ms^{-1}, The coefficient of restitution is e and the particle rebounds in a direction parallel to BA. Find α.



Let **i**,**j** be unit vectors parallel and perpendicular to AB

Velocity of particle before impact with AB is $\begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$

And hence, after this impact it will be $\begin{pmatrix} u\cos\alpha \\ -eu\sin\alpha \end{pmatrix}$

Which is thus the velocity of impact with BC

$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ is a vector parallel to BC and so $\begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}$

is normal to BC. Since the particle rebounds parallel to BA

the velocity after impact with BC may be written as $\begin{pmatrix} -x \\ 0 \end{pmatrix}$

Thus by conservation of momentum parallel to BC

$\begin{pmatrix} -x \\ 0 \end{pmatrix} . \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} u\cos\alpha \\ -eu\sin\alpha \end{pmatrix} . \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \Rightarrow -x\cos\alpha = u(\cos^2\alpha - e\sin^2\alpha.$

and by Newton's law of restitution $\begin{pmatrix} -x \\ 0 \end{pmatrix} . \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} = -e\begin{pmatrix} u\cos\alpha \\ -eu\sin\alpha \end{pmatrix} . \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$

$\Rightarrow -x\sin\alpha = ue(1+e)\sin\alpha\cos\alpha \Rightarrow x = -ue(1+e)\cos\alpha$

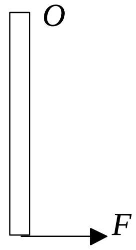
Hence, $ue(1+e)\cos^2\alpha = u(\cos^2\alpha - e\sin^2\alpha) \Rightarrow e\sin^2\alpha = (e^2+e+1)\cos^2\alpha \Rightarrow \tan\alpha = \sqrt{\frac{e^2+e+1}{e}}$

General Statics

So far we have limited our study to that of a single particle, or of a body that may be treated as a single particle. However we are often concerned with the behaviour of collections of particles of various kinds. The simplest such collection with which we are mainly concerned, is one in which the distance between each pair of particles remains constant. We call this a RIGID BODY.

The Moment of a force

It is a matter of common experience that the effect of a force on a large object will depend on the line of action of that force. In particular, as well as tending to move the object linearly there will also usually be a tendency for the object to rotate. It is this TURNING EFFECT that we now investigate.



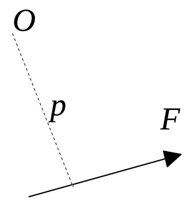
Suppose a rod is freely hinged at one end *O* and hangs freely. If a force is now applied at the other end in a direction perpendicular to the rod then this force will cause the rod to move from the vertical, i.e. to turn about *O*.

Experiments show that this turning effect can be altered by

(a) changing the magnitude of the force.

(b) altering the direction of the force.

(c) moving the point of application nearer to *O*.



The measure of this turning effect is called the MOMENT of the force about *O*. And is defined as the product of the magnitude of the force and the perpendicular distance from *O* of its line of action.

i.e. Moment about *O* of *F* =*pF*

Strictly speaking, "moment about *O*" should read "moment about an axis through *O* perpendicular to the plane containing *O* and *F*." So we need to know the plane in which a force is acting as well as its magnitude and line of action, in order to define its moment. It should be no surprise therefore , that the moment of a force is a vector quantity though the distinction will not be required in M2 or M3.

Equivalent systems of forces.

Sometimes we find that two apparently different systems of forces have

(a) their vector sums equal, i.e. The same resultant force.

(b) the sum of the moments of the forces about any chosen point are the same

We then say that the systems are equivalent.

Couples

Equal and opposite forces *F*, acting at a perpendicular distance *d* apart, are said to form a COUPLE. A couple is completely specified by the product of one of the forces and the distance between them (i.e. by its moment *Fd*)

Equilibrium

Any system of particles, which may or may not comprise a rigid body, is said to be in EQUILIBRIUM if it can remain at rest under the action of the forces acting upon it.. For a rigid body, a necessary and sufficient condition for equilibrium is that:

(a) the vector sum of the external forces should be zero and

(b) the total sum of the moments of the external forces about every point is zero.

In practice, two equivalent sets of conditions may be used:

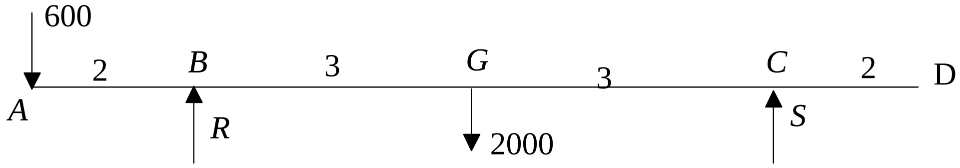
(1) (a) the sum of the resolved parts of the external forces in any two nom-parallel directions must be zero and

(b) the sum of the moments about any one point must be zero.

OR

(2) the sum of the moments about any three non-collinear points must be zero.

Ex. A uniform beam of length 10 m and weight 2 000 N has a man of weight 600 N standing on one end. If the beam is supported at two points 2 m from each end, find the reactions at these supports.



Taking moments about C $6R = 8\times 600 + 3\times 2\,000 = 10\,800 \Rightarrow R = 1800$

and about B $2\times 600 + 6S = 3\times 2000 \Rightarrow S = 800$

So the reactions are 1800 N and 800 N.

Note that we can check this answer by noting that we must have $R + S = 600 + 2000$ ∫

Ex. A ladder of negligible weight and length 8 m is inclined at an angle of 70^o to the horizontal, resting on rough horizontal ground and against a smooth vertical wall. If a man climbs this ladder and it begins to slip when he is three-quarters of the way up, find the coefficient of friction at the ground.

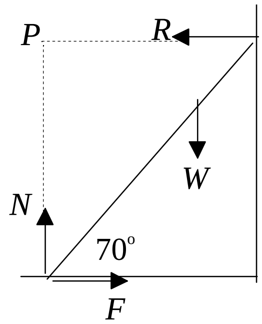
Let the man's weight be W, then with the notation in the diagram.

Resolving vertically $N = W$

Taking moments about P $F\times 8\sin 70^o = W\times 6\cos 70^o \Rightarrow F = \frac{3}{4}\cot 70^o$

Hence, $\frac{F}{N} = \frac{3}{4}\cot 70^o = 0.27$

So the coefficient of friction is 0.27



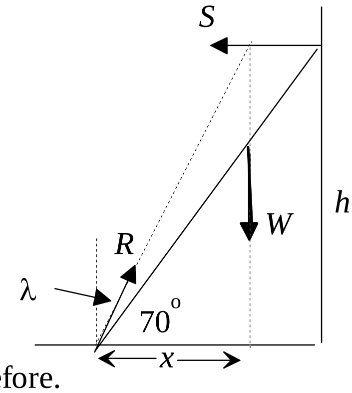
Alternatively . We may solve this problem using the 3-force condition.

"If 3 forces maintain a body in equilibrium then they must be (i) coplanar and (ii) either parallel or concurrent

Combining the normal reaction and the friction force at the ground into a single total reaction we have a 3-force problem and so may use a triangle of forces approach.

Since the three forces are clearly not parallel they must be concurrent

As the man climbs the ladder we can see that the total reaction R makes an increasing angle to the vertical until, when he is three quarters of the way up, this angle must become the angle of friction, and the ladder is on the point of slipping.



Thus, with the notation in the diagram

$h = 8\sin 70^o$ and $x = 6\cos 70^o \Rightarrow \tan\lambda = \frac{6\cos 70^o}{8\sin 70^o} = \frac{3}{4}\cot 70^o$ as before.

Ex. A cylinder of radius a is fixed in contact with horizontal ground and a vertical wall with its axis horizontal and parallel with the wall. A uniform rod of length $2a$ and weight W rests across the cylinder at right angles to its axis with one end in contact with the wall., at an angle θ to the vertical. Find the coefficient of friction between the cylinder and the rod if the wall is smooth and the rod is in limiting equilibrium. (i.e. just about to slip)

With the notation in the diagram

$AC = AD = a\cot\frac{1}{2}\theta$

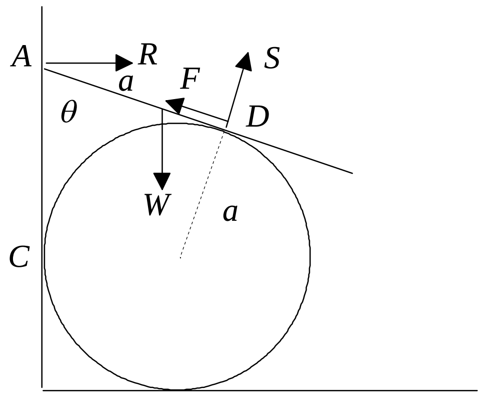
Taking moments about A, $Sa\cot\frac{1}{2}\theta - Wa\sin\theta = 0$

$\Rightarrow S = W\sin\theta\tan\frac{1}{2}\theta = 2W\sin^2\frac{1}{2}\theta$

Resolving vertically $F\cos\theta + S\sin\theta - W = 0$ and $F = \mu S$

$\Rightarrow \mu = \frac{W - S\sin\theta}{S\cos\theta}$

So eliminating S, $\mu = \frac{1 - 2\sin\theta\sin^2\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}\cos\theta$



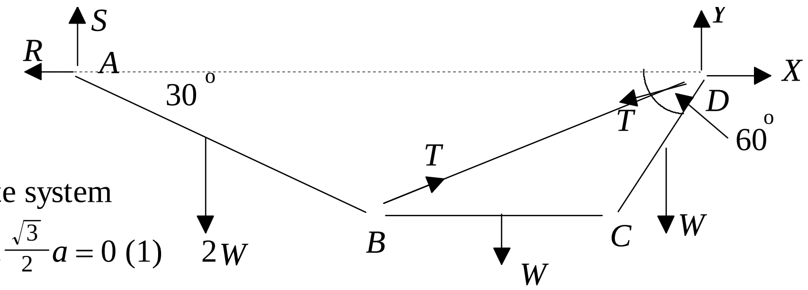
Problems involving more than one body

The usual procedure is to draw diagrams showing the forces acting on each body. This can often be made clearer by drawing a diagram with small gaps between the different bodies as in the following examples. In addition to the external forces acting on the system we must also take into account:

(i) forces at hinges and joints (ii) normal contact and friction forces between bodies in contact.

In each case, equal and opposite forces act on the bodies according to Newton's third law.

Ex. Uniform rods $AB.BC$ and CD of lengths $6a$, $2\sqrt{3}\,a$ and $2\sqrt{3}\,a$ and weights $2W, W$ and W respectively, are smoothly hinged together at B and C. The ends A and D are smoothly hinged to fixed points in the same horizontal line. The system is in equilibrium with AB at 30^o to the horizontal, BC horizontal and DC at 60^o to the horizontal, the position being maintained by a light string joining B and D. Find the tension in this string.



Since $BC = CD$, ΔBCD is isosceles

so $D\hat{B}C = 30^o$

With the notation in the diagram

Taking moments about D for complete system

$R.6\sqrt{3}\,a - 2W.\frac{9}{2}\sqrt{3}\,a - W.2\sqrt{3}\,a - W.\frac{\sqrt{3}}{2}a = 0$ (1)

Taking moments about B for AB

$S.3\sqrt{3}\,a - R.3a - 2W.\frac{3\sqrt{3}}{2}a = 0$ (2) so from (1) and (2) $R = \frac{11\sqrt{3}}{2}W$ (3)

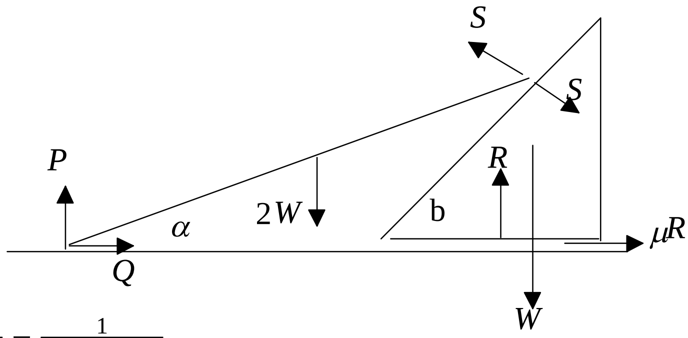
Taking moments about C for AB and BC $S.5\sqrt{3}\,a - R.3a - 2W.\frac{7\sqrt{3}}{2}a + T.\sqrt{3}\,a - W\sqrt{3}\,a = 0$ (4)

(1), (3) and (4) $\Rightarrow \frac{115}{12}W - \frac{33}{12}W - 7W\_T - W = 0 \Rightarrow T = \frac{7}{6}W$

Note that by taking moments about a specific point, forces acting at that point have no effect.

Ex. A uniform rod AB of length $2a$ and weight $2W$ is smoothly hinged at A to a point on a horizontal plane. The rod is in equilibrium at an angle α to the horizontal with the end B resting on the smooth face of a wedge of weight W which has its base in contact with the horizontal plane. The face of the wedge is at an angle β to the horizontal and the rod and a line of greatest slope of the face lie in a vertical plane passing through the centre of mass of the wedge. If the wedge is on the point of slipping, find, in terms of α and β, the coefficient of friction between the wedge and the horizontal plane.

With the notation in the diagram



Moments about A for the rod

$S.2a\cos(\beta-\alpha) - 2Wa\cos\alpha = 0 \Rightarrow S = \frac{W\cos\alpha}{\cos(\beta-\alpha)}$

$\Re(\rightarrow)$ for wedge $S\sin\beta = \mu R$

$\Re(\uparrow)$ for wedge $R = W + S\cos\beta$

Hence, $R = \frac{W(\cos(\beta-\alpha)+\cos\alpha\cos\beta)}{\cos(\beta-\alpha)}$ and $\mu R = \frac{W\cos\alpha\cos\beta}{\cos(\beta-\alpha)}$

So $\mu = \frac{\cos\alpha\cos\beta}{\cos(\beta-\alpha)+\cos\alpha\cos\beta} = \frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta+\sin\alpha+\sin\beta+\cos\alpha\cos\beta} = \frac{1}{2+\tan\alpha\tan\beta}$

Centre of mass

The centre of mass of a system of particles, of masses $m_1,\ m_2,\ m_3$ etc at points whose position vectors are $\mathbf{r}_1,\ \mathbf{r}_2, \mathbf{r}_3$ respectively is given by $\mathbf{r} = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_{23}+m_3\mathbf{r}_3+...}{m_1+m_2+m_3+...}$ i.e. $\mathbf{r} = \sum m\mathbf{r} / \sum m$

Its position being independent of the choice of origin. For a system of particles constituting a rigid body this point coincides with the "centre of gravity", the point through which the resultant of the gravitational forces acting on all the particles may be taken to act.

If a system of particles is coplanar, the centre of mass will lie in the same plane.

If the system of particles is collinear then the centre of mass will lie in that line.

If the system of particles is symmetrical then the centre of mass must lie on the line of symmetry and so if there is more than one line of symmetry the centre of mass must lie at their point of intersection

A uniform body is a body of uniform density.

A lamina is a plane figure of negligible thickness.

Special cases.

A thin uniform rod. The centre of mass is at its mid-point.

A uniform triangular lamina. The centre of mass is at the point of intersections of the medians.

This point is $\frac{2}{3}$ of the way along any median from the corresponding vertex of the triangle

Composite bodies.

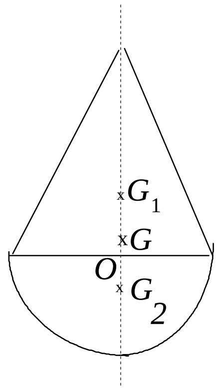
Consider a body consists of 2 or more standard shapes with masses $m_1, m_2, \ldots$and centres of mass at $\mathbf{r}_1$, $\mathbf{r}_2, \ldots$then the centre of mass of the composite body is at $\mathbf{r} = \frac{m_1\mathbf{r}_1 + m\ \ \mathbf{r}_2}{m_1+m_2}$

Remember that there are a lot of standard results on the formula sheet. Familiarise yourself with these.

Ex. A uniform solid consists of a right circular cone of radius a and height $2a$ joined at its base to the base of a hemisphere of radius a. The hemisphere has the same axis of symmetry as the cone. Find the position of the centre of mass of the solid.

The mass of the hemisphere is $\frac{2}{3}\pi a^3\rho$ where ρ is the density.

The mass of the cone is $\frac{1}{3}\pi a^2.2a\rho = \frac{2}{3}\pi a^3\rho$ so the hemisphere and cone have equal mass so we may denote this by m The centre of mass will obviously lie on the axis of symmetry.



So suppose it is a distance x from the common circular base as shown.

Let O be the centre of the common circular plane.

Let G_1, G_2 and G be the centres of mass of the cone, hemisphere and composite solid respectively. Then from standard results

$OG_1 = \frac{1}{4}.2a = \frac{1}{2}a$ and $OG_2 = \frac{3}{8}a$ so if $OG = x$ we will have

$x = \frac{\frac{1}{2}am - \frac{3}{8}am}{m+m} = \frac{1}{16}a$ i.e. the centre of mass lies in the cone a distance $\frac{a}{16}$ from the centre of the common circular face.

Note that in the calculation we had to take one of the distances as negative because they were on opposite sides of the plane we were using as a reference.

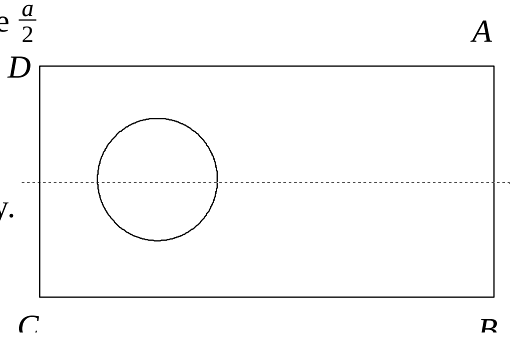
Ex. Masses of 1,2,5 and 2 kg lie in a plane XOY at points with coordinates $(0,1)$, $(0,-1)$, $(1,2)$ and $(2,2)$ respectively.Find the coordinates of their centre of mass.

Let the centre of mass be at $(\bar{x}, \bar{y})$ then taking moments about OY and OX we have $\bar{x} = \sum(m_i x_i)/\sum m_i$ and $\bar{y} = \sum m_i y_i/\sum m_i$ i.e. $\bar{x} = \frac{1\times0+2\times0+5\times1+2\times2}{1+2+5+2} = \frac{9}{10}$ and $\bar{y} = \frac{1\times1+2\times(-1)+5\times2+2\times2}{1+2+5+2} = \frac{13}{10}$

so centre of mass is at the point with coordinates $\left(\frac{9}{10}, \frac{13}{10}\right)$

Ex. A lamina in the shape of a rectangle of length $2a$ and width a has a circle of radius $\frac{a}{4}$ removed

The centre of the circle is on an axis of symmetry and a distance $\frac{a}{2}$ from one of the shorter ends as shown



If the resulting lamina is freely suspended from point A,

Find the angle of AD to the vertical.

Centre of mass of lamina will lie on the line of symmetry.

Let its distance from DC be x

Area of lamina is $2a^2 - \pi\left(\frac{a}{4}\right)^2 = \left(2 - \frac{\pi}{16}\right)a^2$

Taking moments about CD, $\frac{\pi}{16}a^2\rho \times \frac{a}{2} + \left(2 - \frac{\pi}{16}\right)a^2\rho x = 2a^3\rho$

Where ρ is the mass per unit area of the lamina

$\Rightarrow x = \frac{2a^3 - \frac{\pi}{32}a^3}{\left(2-\frac{\pi}{16}\right)a^2} = \frac{(64-\pi)a^3}{32} \times \frac{16}{(32-\pi)a^2} = \frac{(64-\pi)}{(64-2\pi)}a = \frac{60.86}{57.72}a = 1.054a$

When suspended from A, the centre of mass must lie on the vertical through A

$\Rightarrow$ angle of AD to the vertical $= \tan^{-1}\left(\frac{\frac{a}{2}}{2a-x}\right) = \tan^{-1}\left(\frac{1}{4-2.108}\right) = 27.9^o$

INDEX

Information given on STEP formula sheet (relevant to M2 & M3).

Motion in a circle

Transverse velocity: $v = r\dot{\theta}$

Transverse acceleration: $\dot{v} = r\ddot{\theta}$

Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r}$

Centres of Mass (for uniform bodies)

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere, radius r : $\frac{3}{8}r$ form the centre

Hemispherical shell, radius r : $\frac{1}{2}r$ from the centre

Circular arc, radius r, angle at centre 2α : $\frac{r\sin\alpha}{\alpha}$ from centre

Sector of a circle, radius r, angle at centre 2α : $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h : $\frac{1}{4}h$ above the base on the line from centre of base to vertex.

Conical shell of height h : $\frac{1}{3}h$ above the base on line from centre of base to vertex.

Joined