Question 5

The surface Ω is the sphere with Cartesian equation

$$(x-1)^2+(y-1)^2+(z-1)^2=1$$

Evaluate the surface integral

$$\oiint_{\Omega} \left[(x+y)\mathbf{i}+\left(x^2+xy\right)\mathbf{j}+z^2\mathbf{k}\right]\cdot\mathbf{dS},$$

where $\mathbf{dS}$ is a unit surface element on Ω.

You may not use the Divergence Theorem in this question.

$$\boxed{\frac{16}{3}\pi}$$

$$\int_{\Omega} \underline{F}\cdot d\underline{S} = \int_{\Omega} (x+y, x^2+xy, z^2)\cdot \hat{\underline{n}}\, dS = \dots$$

- MOVING THE ORIGIN DOES NOT AFFECT THE ANSWER, SO TRANSLATE THE ORIGIN AT $(1,1,1)$
- THUS $(x-1)^2+(y-1)^2+(z-1)^2=1$ BECOMES $x^2+y^2+z^2=1$

  $(x+y, x^2+xy, z^2)$ BECOMES $\left[(x+1)+(y+1), (x+1)^2+(x+1)(y+1), (z+1)^2\right]$

  TIDY UP

  $= \left[x+y+2, x^2+2x+1+xy+x+y+1, z^2+2z+1\right]$

  $= \left[x+y+2, x^2+xy+3x+y+2, z^2+2z+1\right]$
- LET $f(x,y,z) = x^2+y^2+z^2-1$

  $\nabla f = (2x, 2y, 2z)$

  $\underline{n} = (x,y,z)$

  $|\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$

  $\underline{n} = \hat{\underline{n}} = (x,y,z)$

$$\dots = \int_{S} (x+y+2, x^2+xy+3x+y+2, z^2+2z+1)\cdot(x,y,z)\, dS$$

$$= \int_{S} x^2+xy+2x+x^2y+xy^2+3xy+y^2+2y+z^3+2z^2+z \, dS$$

NOW THE DOMAIN (SURFACE) IS SYMMETRICAL IN x, IN y AND IN z (THINKING AS CIRCULAR CROSS SECTIONS) — SO ALL ODD POWERS IN ANY VARIABLE WILL HAVE NO CONTRIBUTION

$$= \int_{S} \cancel{z^3} + x^2+y^2+2z^2 + \cancel{4xy} + \cancel{x^2y} + \cancel{xy^2} + \cancel{2x} + \cancel{2y} + \cancel{z}\, dS$$

$$= \int_{S} (x^2+y^2+z^2)+z^2 \, dS$$

$$= \int_{S} 1+z^2 \, dS$$

- SWITCH INTO SPHERICAL POLARS

  $x = 1\sin\theta\cos\phi$, $y = 1\sin\theta\sin\phi$, $z = 1\cos\theta$

  $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$

  $\boxed{dS = \sin\theta\, d\theta\, d\phi}$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} (1+\cos^2\theta)(\sin\theta\, d\theta\, d\phi)$$

$$= \left[\int_{\phi=0}^{2\pi} 1\, d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta + \cos^2\theta\sin\theta\, d\theta\right]$$

$$= 2\pi \times \left[-\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= 2\pi \times \left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{\pi}^{0}$$

$$= 2\pi \times \left[\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)\right]$$

$$= 2\pi \times \frac{8}{3}$$

$$= \frac{16}{3}\pi$$