

Question 5

The surface Ω is the sphere with Cartesian equation

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$$

Evaluate the surface integral

$$\oiint_{\Omega} \left[(x+y)\mathbf{i} + (x^2+xy)\mathbf{j} + z^2\mathbf{k} \right] \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is a unit surface element on Ω .

You may not use the Divergence Theorem in this question.

$$\frac{16}{3}\pi$$

The image shows two pages of handwritten work for solving the surface integral. The left page uses a direct method by expanding the vector field components and integrating over the sphere's surface. The right page uses spherical coordinates to simplify the integration.

Left Page Solution:

- Starts with $\int_{\Omega} \mathbf{F} \cdot d\mathbf{S} = \int_{\Omega} (x+y, x^2+xy, z^2) \cdot \hat{n} dS = \dots$
- Notes: "MAJOR: THE ORIGIN DOES NOT AFFECT THE ANSWER, SO TRANSLATE THE ORIGIN AT $(1,1,1)$ "
- Then: $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1 \Rightarrow x^2 + y^2 + z^2 = 1$
- Calculates $\nabla \phi = (2x, 2y, 2z)$ and $|\nabla \phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2$
- Unit normal $\hat{n} = (x, y, z)$
- Integrates $\int_{\Omega} (x+y, x^2+xy, z^2) \cdot (x, y, z) dS$
- Expands to $\int_{\Omega} (x^2 + y^2 + z^2 + xy + yx + z^2) dS$
- Notes: "NOW THE ORIGIN (ORIGIN) IS SIGNIFICANT IN x, y AND IN z (THINKING AS ORIGIN CROSS SECTION) - SO ALL THE POINTS IN THE VOLUME WILL HAVE AN ANGLE OF 120° "

Right Page Solution:

- Starts with $\int_{\Omega} (x+y, x^2+xy, z^2) \cdot \hat{n} dS$
- Then: $\int_{\Omega} (x^2 + y^2 + z^2 + xy + yx + z^2) dS$
- Integrates $\int_{\Omega} (1 + z^2) dS$
- Switchs into spherical coordinates: $x = \sin\theta \cos\phi, y = \sin\theta \sin\phi, z = \cos\theta$
- Unit normal $d\mathbf{S} = \sin\theta d\theta d\phi$
- Integrates $\int_0^\pi \int_0^{2\pi} (1 + \cos^2\theta) \sin\theta d\theta d\phi$
- Calculates $\int_0^\pi (1 + \cos^2\theta) \sin\theta d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^\pi = 2$
- Integrates $\int_0^{2\pi} 2 d\phi = 4\pi$
- Final result: $\frac{16}{3}\pi$