QUESTION PART REFERENCE

Core 4 June 2015

1(a) Express $f(x) = \frac{(19x-2)}{(5-x)(1+6x)}$ in the form $f(x) = \frac{A}{5-x} + \frac{B}{1+6x}$

$19x - 2 = A(1+6x) + B(5-x)$

Let $x = 5$ $\therefore$ $19(5) - 2 = A(1+6(5)) + B(0)$

$93 = 31A$

$\therefore \underline{A = 3}$

Let $x = \frac{-1}{6}$ $\therefore$ $19(\frac{-1}{6}) - 2 = A(0) + B(5 - (-\frac{1}{6}))$

$-\frac{31}{6} = \frac{31}{6}B$

$\therefore \underline{B = -1}$

$\therefore f(x) = \frac{3}{5-x} - \frac{1}{1+6x}$

(b) Find $\int_0^4 f(x)\,dx$. Show that it equals $p\ln 5$ where p is rational.

$\int_0^4 \frac{3}{5-x} - \frac{1}{1+6x}\,dx = \left[-3\ln|5-x| - \frac{1}{6}\ln|1+6x|\right]_0^4$

$\left(\underbrace{-3\ln 1}_{=0} - \frac{1}{6}\ln 25\right) - \left(-3\ln 5 \underbrace{- \frac{1}{6}\ln 1}_{=0}\right)$

$\therefore -\frac{1}{6}\ln 25 + 3\ln 5$

$\Rightarrow -\frac{1}{6}\ln 5^2 + 3\ln 5 \Rightarrow 2\left(\frac{-1}{6}\right)\ln 5 + 3\ln 5$

$\therefore \underline{\frac{8}{3}\ln 5}$

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2(a) Express $2\cos x - 5\sin x$ in the form $R\cos(x+\alpha)$, where $R>0$, $0<\alpha<\pi/2$, giving your value of α to 3s.f.

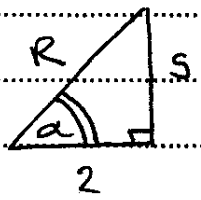
$$2\cos x - 5\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$$

$\therefore$ $R\sin\alpha = 5$ $\qquad \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$R\cos\alpha = 2$

$$\therefore \tan\alpha = \frac{5}{2}$$

$$\tan^{-1}\left(\frac{5}{2}\right) = \alpha = 1.19^c$$



$$R = \sqrt{2^2+5^2} = \underline{\sqrt{29}}$$

$$\therefore \underline{\underline{2\cos x - 5\sin x = \sqrt{29}\cos(x+1.19^c)}}$$

(b)(i) Find the value of x where $2\cos x - 5\sin x$ is a maximum in the value $0<x<2\pi$

Maximum at $\sqrt{29}\cos(x+1.19^c) = \sqrt{29}$

$$\therefore \cos(x+1.19^c) = 1$$

$$\cos^{-1}(1) = x+1.19^c = 0, 2\pi$$

$$\therefore x = -1.19^c, 5.09^c$$

$\therefore$ $\underline{\underline{x = 5.09^c}}$ (within range $0<x<2\pi$)

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2(b) (ii) Solve the equation $2\cos x - 5\sin x + 1 = 0$ in the interval $0 < x < 2\pi$, giving your solutions to 3s.f.

$\therefore \sqrt{29}\cos(x + 1.19) = -1$

$\therefore \cos(x+1.19) = \frac{-1}{\sqrt{29}}$

$\cos^{-1}\left(\frac{-1}{\sqrt{29}}\right) = x + 1.19 = 1.7576\ldots, 2\pi - 1.7576\ldots$

$\therefore x = 0.568^c, 3.34^c$

3(a) $f(x) = 8x^3 - 12x^2 - 2x + d$. Given that when $f(x)$ is divided by $(2x+1)$ the remainder is -2 find d.

$f\left(-\frac{1}{2}\right) = 8\left(-\frac{1}{2}\right)^3 - 12\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + d = -2$

$\therefore -3 + d = -2$

$d = 1$

(b) $g(x) = 8x^3 - 12x^2 - 2x + 3$. Given that $x = -\frac{1}{2}$ is a solution of $g(x) = 0$, express $g(x)$ as a product of 3 linear factors.

$$
\begin{array}{r}
4x^2 - 8x + 3 \\
2x + 1 \overline{\smash{)}\,8x^3 - 12x^2 - 2x + 3} \\
-\underline{(8x^3 + 4x^2 + 0x + 0)} \\
-16x^2 - 2x + 3 \\
-\underline{(-16x^2 - 8x + 0)} \\
6x + 3 \\
-\underline{(6x + 3)} \\
0
\end{array}
$$

$g(x) = (2x+1)(4x^2 - 8x + 3)$

$4x^2 - 8x + 3 = (2x-3)(2x-1)$

$\therefore g(x) = (2x+1)(2x-3)(2x-1)$

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(c) h(x) is given by $\frac{4x^2-1}{g(x)}$. Simplify h(x), showing that it is a decreasing function $(x > 2)$

$$4x^2-1 = (2x+1)(2x-1)$$

$$\therefore \frac{(2x+1)(2x-1)}{(2x+1)(2x-3)(2x-1)} = \frac{1}{2x-3}$$

$$\therefore h(x) = (2x-3)^{-1}$$
$$h'(x) = -2(2x-3)^{-2}$$

$x>2 \therefore h'(x) < 0 \therefore$ decreasing function

4(a) Find the binomial expansion of $(1+5x)^{\frac{1}{5}}$ up to x^2.

$x = 5x, \quad n = \frac{1}{5}$

$$\therefore (1+5x)^{\frac{1}{5}} = 1 + \left(\frac{1}{5}\right)(5x) + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)(5x)^2}{2\times1}$$

$$= 1 + x - 2x^2$$

(b)(i) Find the binomial expansion of $(8-3x)^{-2/3}$ up to x^2.

$$(8-3x)^{-2/3} \Rightarrow \left[8\left(1-\frac{3}{8}x\right)\right]^{-2/3} \Rightarrow \frac{1}{4}\left(1-\frac{3}{8}x\right)^{-2/3}$$

$x = -3/8x, \quad n = -2/3$

$$\therefore (8-3x)^{-2/3} = \frac{1}{4}\left\{1 + \left(\frac{-2}{3}\right)\left(\frac{-3}{8}x\right) + \frac{\left(\frac{-2}{3}\right)\left(\frac{-2}{3}-1\right)\left(\frac{-3}{8}x\right)^2}{2\times1}\right\}$$

$$= \frac{1}{4} + \frac{1}{16}x + \frac{5}{256}x^2$$

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4(b) (ii) Find an estimate for $\sqrt[3]{\frac{1}{81}}$ using your answer to part b(i)

$\sqrt[3]{\frac{1}{81}} = (81)^{-3} \quad \therefore \quad (9)^{\frac{2}{-3}}$

$(8-3x)^{-2/3}$ Let $x = \frac{-1}{3}$

$\therefore \frac{1}{4} + \frac{1}{16}\left(-\frac{1}{3}\right) + \frac{5}{256}\left(-\frac{1}{3}\right)^2$

So $\sqrt[3]{\frac{1}{81}} \approx \underline{\underline{0.2313}}$ (to 4d.p.)

5. A curve is defined by parametric equations $x = \cos 2t$, $y = \sin t$
At the point P, $t = \frac{\pi}{6}$

(a) Find the gradient at P.

$x = \cos 2t \qquad y = \sin t$

$\therefore \frac{dx}{dt} = -2\sin 2t \qquad \therefore \frac{dy}{dt} = \cos t$

$\therefore \frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = \frac{\cos t}{-2\sin 2t}$

At $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{\cos(\pi/6)}{-2\sin\left(\frac{2\pi}{6}\right)} = \underline{\underline{\frac{-1}{2}}}$

(b) Find the equation of the normal to the curve at P in the form $y = mx + c$

At P gradient $= \frac{-1}{2}$ $\therefore$ gradient of normal $= 2$

$y - y_1 = m(x - x_1) \qquad x = \cos\left(\frac{2\pi}{6}\right) = \frac{1}{2}$, $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$y - \frac{1}{2} = 2(x - \frac{1}{2}) \quad \therefore \quad \underline{\underline{y = 2x - \frac{1}{2}}}$

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(c) The normal at P cuts the curve at Q $(\cos 2q, \sin q)$. Set up a quadratic equation in $\sin q$ to find where the normal cuts the curve again and find the x-coordinate of Q.

$y = 2x - \frac{1}{2}$ $\therefore$ $\sin q = 2\cos 2q - \frac{1}{2}$

$\boxed{\cos 2q = \cos^2 q - \sin^2 q}$ $\boxed{\sin^2 q + \cos^2 q = 1}$

$\therefore 2\cos 2q = 2((1-\sin^2 q) - \sin^2 q)$

$= 2 - 4\sin^2 q$

$\therefore -4\sin^2 q + \sin q - \frac{3}{2} = 0$

$(\times 2)$ $8\sin^2 q + 2\sin q - 3 = 0$

$(2\sin q - 1)(4\sin q + 3) = 0$

$\therefore \sin q = \frac{1}{2}, \frac{-3}{4}$

$\therefore q = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ AND $q = \sin^{-1}\left(\frac{-3}{4}\right) = -0.848062079$

$\therefore \cos 2q$ is the x-coordinate at Q $= \cos(2(-0.848\ldots))$

$\therefore$ x-coordinate at Q $= -\frac{1}{8}$

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6 (a) The points A and B have coordinates (3, 2, 10) and (5, −2, 4) respectively.
The line l has equation $\underline{r} = \begin{pmatrix}3\\2\\10\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$

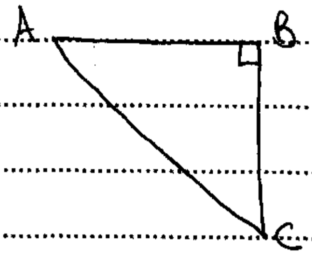
Find the angle between the line l and $\overrightarrow{AB}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}5\\-2\\4\end{pmatrix} - \begin{pmatrix}3\\2\\10\end{pmatrix} = \begin{pmatrix}2\\-4\\-6\end{pmatrix}$$

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} \Rightarrow \frac{\begin{pmatrix}2\\-4\\-6\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}}{\sqrt{2^2+4^2+6^2}\times\sqrt{3^2+1^2+2^2}} = \frac{14}{28} = \frac{1}{2}$$

$$\underline{\theta = \cos^{-1}\left(\tfrac{1}{2}\right) = 60^\circ}$$

(b) The point C lies on the line l such that the angle ABC is 90°.
Find the coordinates of the point C.



$$\overrightarrow{AB}\cdot\overrightarrow{CB} = 0$$

$$\overrightarrow{CB} = \underline{b} - \underline{c} = \begin{pmatrix}5\\-2\\4\end{pmatrix} - \begin{pmatrix}3+3\lambda\\2+\lambda\\10-2\lambda\end{pmatrix}$$

$$\overrightarrow{CB} = \begin{pmatrix}2-3\lambda\\-4-\lambda\\-6+2\lambda\end{pmatrix}$$

$$\overrightarrow{AB}\cdot\overrightarrow{CB} = \begin{pmatrix}2\\-4\\-6\end{pmatrix}\cdot\begin{pmatrix}2-3\lambda\\-4-\lambda\\-6+2\lambda\end{pmatrix} = 0$$

$$\therefore\ 2(2-3\lambda) - 4(-4-\lambda) - 6(-6+2\lambda) = 0$$

$$4 - 6\lambda + 16 + 4\lambda + 36 - 12\lambda = 0$$

$$56 - 14\lambda = 0$$

$$\therefore\ \lambda = 4 \quad \therefore\ \underline{C = (15, 6, 2)}$$

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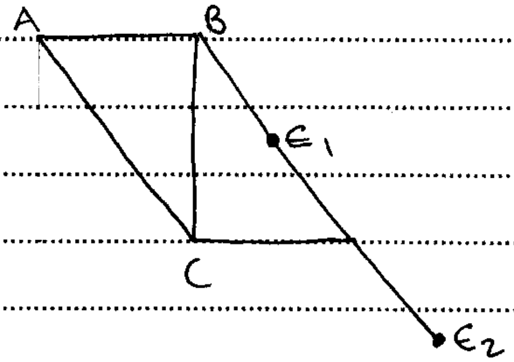
Answer space for question 4

(c) The point D is such that BD is parallel to AC. The point E is such that DE is half that of BD. Find the coordinates of the 2 possible points of E



$\overrightarrow{OE} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{AC}$ $\quad\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix}15\\6\\2\end{pmatrix} - \begin{pmatrix}3\\2\\10\end{pmatrix} = \begin{pmatrix}12\\4\\-8\end{pmatrix}$

$\Rightarrow \begin{pmatrix}5\\-2\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}12\\4\\-8\end{pmatrix}$

$= \begin{pmatrix}11\\0\\0\end{pmatrix} \therefore E_1 = (11, 0, 0)$

or $\overrightarrow{OE} = \overrightarrow{OB} + \frac{3}{2}\overrightarrow{AC}$

$\Rightarrow \begin{pmatrix}5\\-2\\4\end{pmatrix} + \frac{3}{2}\begin{pmatrix}12\\4\\-8\end{pmatrix}$

$= \begin{pmatrix}23\\4\\-8\end{pmatrix} \therefore E_2 = (23, 4, -8)$

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7. $y^3 + 2e^{-3x}y - x = k$

(a) The point $P(\ln 2, \frac{1}{2})$ lies on the curve. Find k where $k = q - \ln 2$ and q is obtained.

$$\left(\frac{1}{2}\right)^3 + 2e^{-3\ln 2}\left(\frac{1}{2}\right) - \ln 2 = k$$

$$\therefore k = \frac{1}{4} - \ln 2$$

(b) Find the gradient at the point P.

$$y^3 \Rightarrow 3y^2\left(\frac{dy}{dx}\right) \qquad -x \Rightarrow -1$$

$2e^{-3x}y$ Product Rule $u = 2e^{-3x}$ $v = y$

$u' = -6e^{-3x}$ $v' = \frac{dy}{dx}$

$$\therefore u'v + uv' = -6ye^{-3x} + 2e^{-3x}\left(\frac{dy}{dx}\right)$$

$$\therefore 3y^2\left(\frac{dy}{dx}\right) - 6ye^{-3x} + 2e^{-3x}\left(\frac{dy}{dx}\right) - 1 = 0$$

$$\therefore \left(3y^2 + 2e^{-3x}\right)\frac{dy}{dx} = 1 + 6ye^{-3x}$$

$$\therefore \frac{dy}{dx} = \frac{1 + 6ye^{-3x}}{3y^2 + 2e^{-3x}}$$

At $(\ln 2, \frac{1}{2})$ $\frac{dy}{dx} = \frac{1 + 6(\frac{1}{2})e^{-3\ln 2}}{3(\frac{1}{2})^2 + 2e^{-3\ln 2}} = \frac{11}{8}$

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Answer space for question 4

8. (a) Initially a pond is empty, the depth of the pond is x at time t. The pond is filled with water according to the differential equation $\frac{dx}{dt} = \frac{\sqrt{4+5x}}{5(1+t)^2}$

Find x as a function of t.

$$\int \frac{5}{\sqrt{4+5x}}\,dx = \int \frac{1}{(1+t)^2}\,dt$$

$$\therefore \int 5(4+5x)^{-1/2}\,dx = \int (1+t)^{-2}\,dt$$

$$\Rightarrow 2(4+5x)^{1/2} = -1(1+t)^{-1} + c$$

At $t=0$, $x=0$

$$\therefore 2(4+5(0))^{1/2} = -1(1+(0))^{-1} + c$$

$$\therefore 4+1 = c$$

$$c = 5$$

$$\therefore 2(4+5x)^{1/2} = -1(1+t)^{-1} + 5$$

$(\div 2)$

$$\therefore (4+5x)^{1/2} = -\frac{1}{2}(1+t)^{-1} + \frac{5}{2}$$

square both sides $\therefore 4+5x = \left(-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right)^2$

(-4) $\therefore 5x = \left(-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right)^2 - 4$

$(\div 5)$ $\therefore x = \frac{1}{5}\left[\left(-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right)^2 - 4\right]$

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8(b)(i) Another pond models a circle of radius r. The rate of change of radius r is inversely proportional to the pond surface area. Write a differential equation to model this situation

$$\frac{dr}{dt} \propto \frac{1}{A} \quad \therefore \frac{dr}{dt} = \frac{k}{\pi r^2}$$

(ii) The rate of change of radius r is 4.5 when the radius is 1 metre. Find the radius when the rate is 0.5

$$\frac{dr}{dt} = \frac{k}{\pi r^2} \quad \therefore 4.5 = \frac{k}{\pi (1)^2} \quad \therefore k = 4.5\pi$$

$$\text{At } \frac{dr}{dt} = 0.5 \quad \Rightarrow 0.5 = \frac{4.5\cancel{\pi}}{\cancel{\pi} r^2} = \frac{4.5}{r^2}$$

$$\therefore r^2 = \frac{4.5}{0.5} = 9$$

$$\therefore r = \pm 3$$

radius must be 3 metres as r must be positive

$$r = 3\text{m}$$