**Question 5**

Carry out the following integrations:

1. $\displaystyle\int \sin 3x \cos 2x \; dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

2. $\displaystyle\int \frac{1}{\sin x \cos x} \; dx = -\frac{1}{2}\ln\left|\operatorname{cosec} 2x + \cot 2x\right| + C = \ln\left|\tan x\right| + C$

3. $\displaystyle\int \frac{1}{1-\sin x} \; dx = \sec x + \tan x + C$

4. $\displaystyle\int \sin^2 2x \; dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

5. $\displaystyle\int \frac{\cos 2x}{\cos^2 x} \; dx = 2x - \tan x + C$

6. $\displaystyle\int \cos^2 x \sin^2 x \; dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

7. $\displaystyle\int (\sin x + 2\cos x)^2 \; dx = \frac{5}{2}x + 2\sin^2 x + \frac{3}{4}\sin 2x + C$

8. $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x} \; dx = -2\cot 2x + C$

9. $\displaystyle\int \sqrt{\sin^2 x + (\cos x - 1)^2} \; dx = -4\cos\left(\frac{x}{2}\right) + C$

10. $\displaystyle\int \frac{1-\cos x}{1+\cos x} \; dx = 2\tan\left(\frac{x}{2}\right) - x + C = -2\cot x - x + 2\operatorname{cosec} x + C$

1. $\displaystyle\int \sin 3x \cos 2x \, dx = \int \tfrac{1}{2}\sin 5x + \tfrac{1}{2}\sin x \, dx = -\tfrac{1}{10}\cos 5x - \tfrac{1}{2}\cos x$

$\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$
$\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$
$\sin 5x + \sin x = 2\sin 3x \cos 2x$ $\quad$ $\therefore \sin 3x \cos 2x = \tfrac{1}{2}\sin 5x + \tfrac{1}{2}\sin x$

2. $\displaystyle\int \frac{1}{\sin x \cos x}\, dx = \int \frac{2}{2\sin x \cos x}\, dx = \int \frac{2}{\sin 2x}\, dx = \int 2\operatorname{cosec} 2x \, dx$

$= \ln|\tan x| + C$

3. $\displaystyle\int \frac{1}{1-\sin x}\, dx = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\, dx = \int \frac{1+\sin x}{1-\sin^2 x}\, dx$

$\displaystyle= \int \frac{1+\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx = \int \sec^2 x + \frac{\sin x}{\cos x}\times\frac{1}{\cos x}\, dx$

$\displaystyle= \int \sec^2 x + \tan x \sec x \, dx = \tan x + \sec x + C$

4. $\displaystyle\int \sin^2 2x \, dx = \int \tfrac{1}{2} - \tfrac{1}{2}\cos 4x \, dx = \tfrac{1}{2}x - \tfrac{1}{8}\sin 4x + C$

5. $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\, dx = \int \frac{2\cos^2 x - 1}{\cos^2 x}\, dx = \int \frac{2\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}\, dx = \int 2 - \sec^2 x \, dx$

$= 2x - \tan x + C$

6. $\displaystyle\int \cos^2 x \sin^2 x \, dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx = \int \tfrac{1}{4} - \tfrac{1}{4}\cos^2 2x \, dx$

$\displaystyle= \int \tfrac{1}{4} - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) dx = \int \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{1}{8}\cos 4x \, dx$

$\displaystyle= \int \tfrac{1}{8} - \tfrac{1}{8}\cos 4x \, dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$

ALTERNATIVE: $\displaystyle= \int (\cos x \sin x)^2 \, dx = \int \left(\tfrac{1}{2}\times 2\sin x \cos x\right)^2 dx = \int \left(\tfrac{1}{2}\sin 2x\right)^2 dx$

$\displaystyle= \int \tfrac{1}{4}\sin^2 2x \, dx = \int \tfrac{1}{4}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right) dx$

$\displaystyle= \int \tfrac{1}{8} - \tfrac{1}{8}\cos 4x \, dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$

7. $\displaystyle\int (\sin x + 2\cos x)^2 \, dx = \int \sin^2 x + 4\sin x \cos x + 4\cos^2 x \, dx$

$\displaystyle= \int \tfrac{1}{2} - \tfrac{1}{2}\cos 2x + 2\sin 2x + 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$

$\displaystyle= \int \tfrac{5}{2} + \tfrac{3}{2}\cos 2x + 2\sin 2x \, dx = \tfrac{5}{2}x + \tfrac{3}{4}\sin 2x - \cos 2x + C$

8. $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\, dx = \int \frac{1}{(\sin x \cos x)^2}\, dx = \int \frac{1}{\left(\tfrac{1}{2}\times 2\sin x \cos x\right)^2}\, dx = \int \frac{1}{\left(\tfrac{1}{2}\sin 2x\right)^2}\, dx$

$\displaystyle= \int \frac{1}{\tfrac{1}{4}\sin^2 2x}\, dx = \int 4\operatorname{cosec}^2 2x \, dx = -2\cot 2x + C$

ALTERNATIVE $\displaystyle= \int \frac{1}{\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)}\, dx = \int \frac{1}{\tfrac{1}{4} - \tfrac{1}{4}\cos^2 2x}\, dx$

$\displaystyle= \int \frac{4}{1-\cos^2 2x}\, dx = \int \frac{4}{\sin^2 2x}\, dx = \int 4\operatorname{cosec}^2 2x$

$= -2\cot 2x + C$

9. $\displaystyle\int \sqrt{\sin^2 x + (\cos x - 1)^2}\, dx = \int \sqrt{\sin^2 x + \cos^2 x - 2\cos x + 1}\, dx$

$\displaystyle= \int \sqrt{1 + 1 - 2\cos x}\, dx = \int \sqrt{2 - 2\cos x}\, dx$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\cos\left(2\times\tfrac{\theta}{2}\right) = 1 - 2\sin^2\left(\tfrac{\theta}{2}\right)$

$\displaystyle= \int \sqrt{2 - 2\left(1 - 2\sin^2\tfrac{x}{2}\right)}\, dx = \int \sqrt{4\sin^2\tfrac{x}{2}}\, dx$

$\displaystyle= \int 2\sin\tfrac{x}{2}\, dx = -4\cos\tfrac{x}{2} + C$

10. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\, dx = \int \frac{(1-\cos x)(1-\cos x)}{(1+\cos x)(1-\cos x)}\, dx = \int \frac{1 - 2\cos x + \cos^2 x}{1-\cos^2 x}\, dx$

$\displaystyle= \int \frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\, dx = \int \operatorname{cosec}^2 x - 2\frac{\cos x}{\sin x}\frac{1}{\sin x} + \cot^2 x \, dx$

$\displaystyle= \int \operatorname{cosec}^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x - 1 \, dx$

$\displaystyle= \int 2\operatorname{cosec}^2 x - 2\cot x \operatorname{cosec} x - 1 \, dx$

$= -2\cot x + 2\operatorname{cosec} x - x + C$

Can also be done by "little t" identities