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1. (a) Find the binomial expansion of

$$(4+5x)^{\frac{1}{2}}, \quad |x| < \frac{4}{5}$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient in its simplest form.

(5)

(b) Find the exact value of $(4+5x)^{\frac{1}{2}}$ when $x = \frac{1}{10}$

Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined.

(1)

(c) Substitute $x = \frac{1}{10}$ into your binomial expansion from part (a) and hence find an approximate value for $\sqrt{2}$

Give your answer in the form $\frac{p}{q}$ where p and q are integers.

(2)

a) $(4+5x)^{1/2}$

$= 2\left(1+\frac{5x}{4}\right)^{1/2}$

$= 2\left[1 + \frac{1}{2}\left(\frac{5x}{4}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{5x}{4}\right)^2}{2!}\right]$

$= 2 + \frac{5x}{4} + - \frac{25x^2}{64}$

b) $x = \frac{1}{10}$

$\left(4+5\left(\frac{1}{10}\right)\right)^{1/2} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}, \quad k = \frac{3}{2}$

c) $\frac{3\sqrt{2}}{2} = 2 + \frac{5}{4}\left(\frac{1}{10}\right) - \frac{25}{64}\left(\frac{1}{10}\right)^2$

$= \frac{543}{256}$

$\sqrt{2} = \frac{181}{128}, \quad p = 181, \quad q = 128$

P 4 4 8 2 7 A 0 2 3 2

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2. The curve C has equation

$$x^2 - 3xy - 4y^2 + 64 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

(b) Find the coordinates of the points on C where $\frac{dy}{dx} = 0$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

a) $2x - 3x\frac{dy}{dx} - 3y + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-3x + 8y) = 3y - 2x$

$\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$

b) $\frac{3y - 2x}{8y - 3x} = 0$

$3y - 2x = 0$

$3y = 2x$

$y = \frac{2x}{3}$

$x^2 - 3x\left(\frac{2x}{3}\right) - 4\left(\frac{2x}{3}\right)^2 + 64 = 0$

$x^2 - 2x^2 - \frac{16x^2}{9} + 64 = 0$

$-\frac{25x^2}{9} + 64 = 0$

$\frac{25x^2}{9} = 64$

$\frac{5x}{3} = \pm 8$

$x = \pm \frac{24}{5}$

$x = \frac{24}{5} \Rightarrow y = \frac{2}{3}\left(\frac{24}{5}\right) = \frac{16}{5}$ $\left(\frac{24}{5}, \frac{16}{5}\right)$

$x = -\frac{24}{5} \Rightarrow y = \frac{2}{3}\left(-\frac{24}{5}\right) = -\frac{16}{5}$ $\left(-\frac{24}{5}, -\frac{16}{5}\right)$

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3.

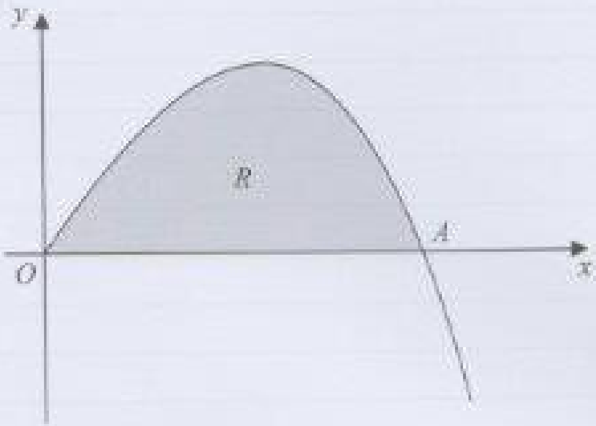

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = 4x - xe^{\frac{1}{2}x}$, $x \geqslant 0$

The curve meets the x-axis at the origin O and cuts the x-axis at the point A.

(a) Find, in terms of ln2, the x coordinate of the point A.

(2)

(b) Find

$$\int xe^{\frac{1}{2}x}\,dx$$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the x-axis and the curve with equation

$$y = 4x - xe^{\frac{1}{2}x}, \ x \geqslant 0$$

(c) Find, by integration, the exact value for the area of R.
Give your answer in terms of ln2

(3)

a) A is when $y=0$

$0 = 4x - xe^{\frac{1}{2}x}$

$= x(4 - e^{\frac{1}{2}x})$

$x \neq 0$ at A so $4 - e^{\frac{1}{2}x} = 0$

$4 = e^{\frac{1}{2}x}$

$\ln 4 = \frac{1}{2}x$

$x = 2\ln 4 = 4\ln 2$

$A(4\ln 2, 0)$

P 4 4 8 2 7 A 0 8 3 2

Question 3 continued

Leave blank

b) $\int x e^{\frac{1}{2}x}\, dx$

Let $u = x$ $\quad \frac{dv}{dx} = e^{\frac{1}{2}x}$

$\frac{du}{dx} = 1$ $\quad v = 2e^{\frac{1}{2}x}$

$= 2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x}\, dx$

$= 2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x} + c$

c) $\int_0^{4\ln 2} 4x - xe^{\frac{1}{2}x}\, dx$

$= \int_0^{4\ln 2} 4x\, dx - \int_0^{4\ln 2} xe^{\frac{1}{2}x}$

$= [2x^2]_0^{4\ln 2} - [2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}]_0^{4\ln 2}$

$= 32(\ln 2)^2 - [8\ln 2\, e^{2\ln 2} - 4e^{2\ln 2} + 4]$

$= 32(\ln 2)^2 - 32\ln 2 + 12$

Leave blank

4. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

where λ and μ are scalar parameters and p is a constant.

The lines l_1 and l_2 intersect at the point A.

(a) Find the coordinates of A.

(2)

(b) Find the value of the constant p.

(3)

(c) Find the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.

(3)

The point B lies on l_2 where $\mu = 1$

(d) Find the shortest distance from the point B to the line l_1, giving your answer to 3 significant figures.

(3)

a) $5 = 8 + 3\mu$

$-3 = 3\mu$

$\mu = -1$

$$A = \begin{pmatrix} 8 + 3(-1) \\ 5 + 4(-1) \\ -2 - 5(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

b) $1 = \lambda - 3$

$\lambda = 4$

$3 = p + (-3)(4)$

$= -12 + p$

$p = 15$

c) $\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = 4 + 15 = 19$

Leave blank

Question 4 continued

$\sqrt{0^2+1^2+(-3)^2} = \sqrt{10}$, $\sqrt{3^2+4^2+(-5)^2} = 5\sqrt{2}$

$\cos\theta = \left| \dfrac{19}{\sqrt{10}\cdot 5\sqrt{2}} \right|$

$= \dfrac{19}{10\sqrt{5}}$

$\theta = 31.82$

d) $\mu = 1$

$B = \begin{pmatrix} 8+3(1) \\ 5+4(1) \\ -2+(-1)(1) \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix}$

$\begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}$

$\sqrt{6^2+8^2+\cancel{}+(-10)^2} = 10\sqrt{2}$

$10\sqrt{2}\sin\theta = 10\sqrt{2}\sin(31.82) = 7.46$

Leave blank

5. A curve C has parametric equations

$$x = 4t + 3, \quad y = 4t + 8 + \frac{5}{2t}, \quad t \neq 0$$

(a) Find the value of $\frac{dy}{dx}$ at the point on C where $t = 2$, giving your answer as a fraction in its simplest form.

(3)

(b) Show that the cartesian equation of the curve C can be written in the form

$$y = \frac{x^2 + ax + b}{x - 3}, \quad x \neq 3$$

where a and b are integers to be determined.

(3)

a) $\frac{dy}{dt} = 4 - \frac{5}{2t^2}$, $\frac{dx}{dt} = 4$

$\frac{dy}{dx} = \frac{1}{4}\left(4 - \frac{5}{2t^2}\right) = 1 - \frac{5}{8t^2}$

when $t = 2$:

$\frac{dy}{dx} = 1 - \frac{5}{8(2)^2} = \frac{27}{32}$

b) $x = 4t + 3$

$t = \frac{x-3}{4}$

$y = 4t + 8 + \frac{5}{2t}$

$= 4\left(\frac{x-3}{4}\right) + 8 + \frac{5}{2\left(\frac{x-3}{4}\right)}$

$= x - 3 + 8 + \frac{10}{x-3}$

$= x + 5 + \frac{10}{x-3}$

$= \frac{(x+5)(x-3) + 10}{x-3}$

$= \frac{x^2 + 2x - 15 + 10}{x-3} = \frac{x^2 + 2x - 5}{x-3}$, $a = 2$, $b = -5$

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6.

Diagram not to scale

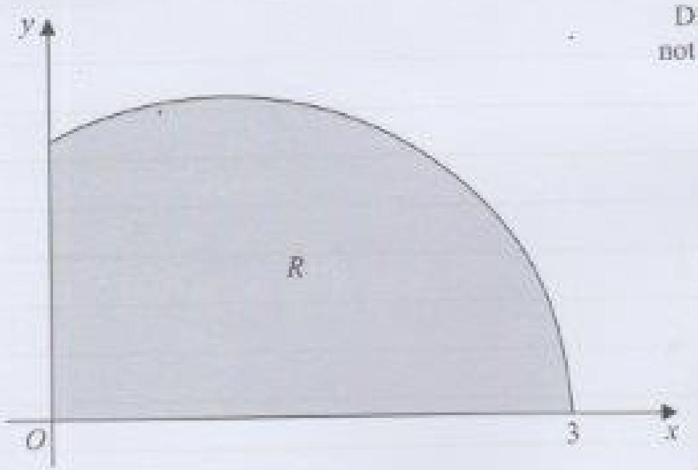

Figure 2

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3-x)(x+1)}$, $0 \leqslant x \leqslant 3$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis, and the y-axis.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that

$$\int_0^3 \sqrt{(3-x)(x+1)}\,\mathrm{d}x = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

where k is a constant to be determined.

(5)

(b) Hence find, by integration, the exact area of R.

(3)

a) $x = 1 + 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$\sqrt{(3-x)(x+1)} = \sqrt{(3-(1+2\sin\theta))(1+2\sin\theta+1)}$

$= \sqrt{(2-2\sin\theta)(2+2\sin\theta)}$

$= \sqrt{4-4\sin^2\theta}$

$= \sqrt{4\cos^2\theta}$

$= 2\cos\theta$

$3 = 1 + 2\sin\theta$ $\quad$ $0 = 1 + 2\sin\theta$

$2 = 2\sin\theta$ $\quad$ $-1 = 2\sin\theta$

$1 = \sin\theta$ $\quad$ $-\frac{1}{2} = \sin\theta$

$\theta = \frac{\pi}{2}$ $\quad$ $\theta = -\frac{\pi}{6}$

Leave blank

Question 6 continued

$$\therefore \int_{-\frac{\pi}{6}}^{\pi/2} 2\cos\theta \cdot 2\cos\theta \, d\theta$$

$$= 4\int_{-\pi/6}^{\pi/2} \cos^2\theta \, d\theta$$

$K = 4$

b) $\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$2\int_{-\pi/6}^{\pi/2} \cos 2\theta + 1 \, d\theta$$

$$= 2\left[\frac{1}{2}\sin 2\theta + \theta\right]_{-\pi/6}^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} + \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right]$$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

Leave blank

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time. Give your answer in years to 3 significant figures.

(3)

a) $\dfrac{2}{P(P-2)} = \dfrac{A}{P} + \dfrac{B}{P-2}$

$2 = A(P-2) + B(P)$

Let $p=2$ $\qquad$ Let $p=0$

$2 = 2B$ $\qquad$ $2 = A(-2)$

$B=1$ $\qquad$ $A=-1$

$\dfrac{1}{P-2} - \dfrac{1}{P}$

b) $\dfrac{dp}{dt} = \dfrac{1}{2}P(P-2)\cos 2t$

$\dfrac{2}{P(P-2)}\,dp = \cos 2t\,dt$

$\displaystyle\int \frac{2}{P(P-2)}\,dp = \int \cos 2t\,dt$

Question 7 continued

$$\int \frac{1}{P-2} - \frac{1}{P}\, dP = \int \cos 2t \, dt$$

$$\ln(P-2) - \ln|P| = \frac{1}{2}\sin 2t + c$$

When $p = 3,\ t = 0$

$$\ln(3-2) - \ln(3) = \frac{1}{2}\sin(2\cdot 0) + c$$

$$\ln(1) - \ln(3) = c$$

$$c = -\ln 3$$

$$\ln(P-2) - \ln(P) = \frac{1}{2}\sin 2t - \ln 3$$

$$\ln\left(\frac{P-2}{P}\right) = \frac{1}{2}\sin 2t - \ln 3$$

$$\ln 3 + \ln\left(1 - \frac{2}{P}\right) = \frac{1}{2}\sin 2t$$

$$\ln\left(3 - \frac{6}{P}\right) = \frac{1}{2}\sin 2t$$

$$3 - \frac{6}{P} = e^{\frac{1}{2}\sin 2t}$$

$$\frac{6}{P} = 3 - e^{\frac{1}{2}\sin 2t}$$

$$\frac{P}{6} = \frac{1}{3 - e^{\frac{1}{2}\sin 2t}}$$

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

c) $\cancel{4000} = \dfrac{6}{3 - e^{\frac{1}{2}\sin 2t}}$

$$3 - e^{\frac{1}{2}\sin 2t} = \frac{6}{\cancel{4000}\ 4} = \cancel{\frac{3}{2000}}\ \frac{3}{2}$$

$$e^{\frac{1}{2}\sin 2t} = \cancel{\frac{5997}{2000}}\ \cancel{\frac{3}{2}}\ \frac{3}{2}$$

$$\frac{1}{2}\sin 2t = \cancel{\ln\left(\frac{5997}{2000}\right)}\ \ln\left(\frac{3}{2}\right)$$

Question 7 continued

$\sin 2t = [illegible]\left(\frac{9994}{10000}\right) 2\ln\left(\frac{3}{2}\right)$

$2t = 0.9437400934$

$t = 0.473$ years

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8.

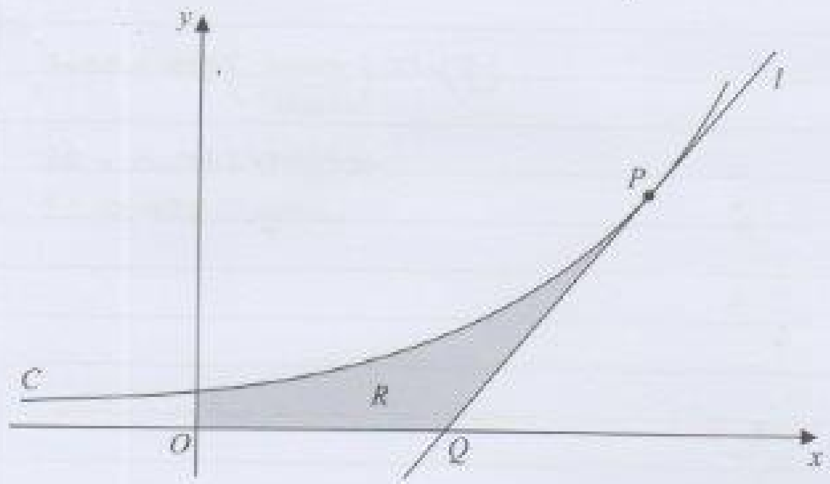

Diagram not to scale

Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = 3^x$$

The point P lies on C and has coordinates $(2, 9)$.

The line l is a tangent to C at P. The line l cuts the x-axis at the point Q.

(a) Find the exact value of the x coordinate of Q.

(4)

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis, the y-axis and the line l. This region R is rotated through 360° about the x-axis.

(b) Use integration to find the exact value of the volume of the solid generated.

Give your answer in the form $\frac{p}{q}$ where p and q are exact constants.

[*You may assume the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(6)

a) $y = 3^x$, $P(2,9)$

$\frac{dy}{dx} = 3^x \ln 3$

$\frac{dy}{dx}$ at $P = 3^2 \ln 3 = 9\ln 3$

$y - 9 = 9\ln 3(x-2)$

at Q, $y=0$

$0 - 9 = 9\ln 3(x-2)$

$-\frac{1}{\ln 3} = x - 2 \implies x = 2 - \frac{1}{\ln 3}$, $Q(2 - \frac{1}{\ln 3}, 0)$

Leave blank

Question 8 continued

b) $\text{volume} = \pi\int_0^2 3^{2x}\,dx = \pi\int_0^2 9^x\,dx$

$= \pi\left[\frac{9^x}{\ln 9}\right]_0^2$

$= \pi\left[\frac{9^2}{\ln 9} - \frac{9^0}{\ln 9}\right]$

$= \pi \cdot \frac{80}{\ln 9}$

$= \frac{40\pi}{\ln 3}$

$\frac{1}{3}\pi(9)^2\left(2-\left(2-\frac{1}{\ln 3}\right)\right)$

$= \frac{1}{3}\pi(81)\left(\frac{1}{\ln 3}\right)$

$= \frac{27\pi}{\ln 3}$

$\frac{40\pi}{\ln 3} - \frac{27\pi}{\ln 3}$

$= \frac{13\pi}{\ln 3}$