Question 70 (*****)

The function f is defined as

$$f(n) = \frac{e^{-\lambda} \lambda^n}{n!},$$

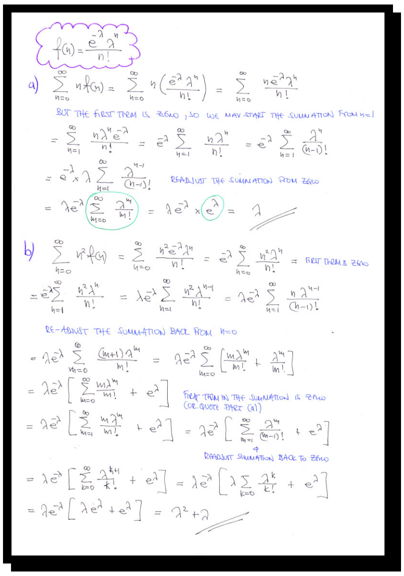
where $n = 0, 1, 2, 3, 4, \ldots$ and λ is a positive constant.

By showing a detailed method, prove that …

a) … $\displaystyle\sum_{n=0}^{\infty} \left[n f(n) \right] = \lambda$.

b) … $\displaystyle\sum_{n=0}^{\infty} \left[n^2 f(n) \right] = \lambda^2 + \lambda$.

$\boxed{\text{proof}}$



$$f(n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

a) $\displaystyle\sum_{n=0}^{\infty} n f(n) = \sum_{n=0}^{\infty} n\left(\frac{e^{-\lambda}\lambda^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{n e^{-\lambda}\lambda^n}{n!}$

BUT THE FIRST TERM IS ZERO, SO WE MAY START THE SUMMATION FROM $n=1$

$\displaystyle = \sum_{n=1}^{\infty} \frac{n\lambda^n e^{-\lambda}}{n!} = e^{-\lambda}\sum_{n=1}^{\infty} \frac{n\lambda^n}{n!} = e^{-\lambda}\sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}$

$\displaystyle = e^{-\lambda} \times \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$ READJUST THE SUMMATION FROM ZERO

$\displaystyle = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} \times e^{\lambda} = \lambda$

b) $\displaystyle\sum_{n=0}^{\infty} n^2 f(n) = \sum_{n=0}^{\infty} \frac{n^2 e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty} \frac{n^2\lambda^n}{n!} =$ FIRST TERM IS ZERO

$\displaystyle = e^{-\lambda}\sum_{n=1}^{\infty} \frac{n^2\lambda^n}{n!} = \lambda e^{-\lambda}\sum_{n=1}^{\infty} \frac{n^2\lambda^{n-1}}{n!} = \lambda e^{-\lambda}\sum_{n=1}^{\infty} \frac{n\lambda^{n-1}}{(n-1)!}$

RE-ADJUST THE SUMMATION BACK FROM $n=0$

$\displaystyle = \lambda e^{-\lambda}\sum_{m=0}^{\infty} \frac{(m+1)\lambda^m}{m!} = \lambda e^{-\lambda}\sum_{m=0}^{\infty}\left[\frac{m\lambda^m}{m!} + \frac{\lambda^m}{m!}\right]$

$\displaystyle = \lambda e^{-\lambda}\left[\sum_{m=0}^{\infty} \frac{m\lambda^m}{m!} + e^{\lambda}\right]$ FIRST TERM IN THE SUMMATION IS ZERO (OR QUOTE PART (a))

$\displaystyle = \lambda e^{-\lambda}\left[\sum_{m=1}^{\infty} \frac{m\lambda^m}{m!} + e^{\lambda}\right] = \lambda e^{-\lambda}\left[\sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} + e^{\lambda}\right]$

READJUST SUMMATION BACK TO ZERO

$\displaystyle = \lambda e^{-\lambda}\left[\sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} + e^{\lambda}\right] = \lambda e^{-\lambda}\left[\lambda\sum_{k=0} \frac{\lambda^k}{k!} + e^{\lambda}\right]$

$= \lambda e^{-\lambda}\left[\lambda e^{\lambda} + e^{\lambda}\right] = \lambda^2 + \lambda$

Question 73 **(*****)**

Consider the binomial infinite series expansion

$$(1+ax)^n,$$

where $a \in \mathbb{R}$, $n \in \mathbb{Q}$, $n \notin \mathbb{N}$.

Show that the series converges if $|ax|<1$.

$\boxed{\text{proof}}$

$(1+ax)^n \quad n \in \mathbb{Q} \quad n \notin \mathbb{N}$

$$(1+ax)^n = 1 + \frac{n}{1!}(ax) + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \dots$$

- GENERAL TERM

$$(1+ax)^n = \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}(ax)^r$$

- THUS FOR CONVERGENCE BY D'ALEMBERT'S RATIO TEST

$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| \to L < 1 \quad \text{FOR CONVERGENCE}$$

- HENCE

$$\left|\frac{\frac{n(n-1)(n-2)\dots(n-(r+1)+1)}{(r+1)!}(ax)^{r+1}}{\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}(ax)^r}\right| = \left|\frac{\frac{n(n-1)(n-2)\dots(n-r)}{(r+1)!}(ax)^{r+1}}{\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}(ax)^r}\right|$$

$$= \left|\frac{\cancel{r!}\,\cancel{n(n-1)(n-2)\dots}(n-r)(ax)^{r+1}}{\cancel{(r+1)!}\,\cancel{n(n-1)(n-2)\dots(n-r+1)}(ax)^r}\right| = \left|\frac{(n-r)}{(r+1)}(ax)\right| = \left|\frac{n-r}{r+1}\right||ax|$$

- $$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\left|\frac{n-r}{r+1}\right||ax| = |ax|$$

TENDS TO -1

$\therefore$ FOR CONVERGENCE $|ax|<1$

Question 75 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6 \ .$$

proof

$S = \sum_{r=1}^{\infty} \frac{r^2}{2^r} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \frac{49}{128} + \dots$

$$\left.\begin{aligned} 2S &= 1 + \tfrac{4}{2} + \tfrac{9}{4} + \tfrac{16}{8} + \tfrac{25}{16} + \tfrac{36}{32} + \tfrac{49}{64} + \dots \\ S &= \tfrac{1}{2} + \tfrac{4}{4} + \tfrac{9}{8} + \tfrac{16}{16} + \tfrac{25}{32} + \tfrac{36}{64} + \dots \end{aligned}\right\} \text{SUBTRACT}$$

$S = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \frac{11}{32} + \frac{13}{64} + \dots$

$S = 1 + \sum_{r=1}^{\infty} \frac{2r+1}{2^r}$

$S = 1 + 2\sum_{r=1}^{\infty} \frac{r}{2^r} + \sum_{r=1}^{\infty} \frac{1}{2^r} \longleftarrow$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ STANDARD G.P $= 1$

$S = 2 + 2\sum_{r=1}^{\infty} \frac{r}{2^r}$

$S = 2 + 2T$

$T = \sum_{r=1}^{\infty} \frac{r}{2^r} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots$

$$\left.\begin{aligned} 2T &= 1 + 1 + \tfrac{3}{4} + \tfrac{4}{8} + \tfrac{5}{16} + \tfrac{6}{32} + \dots \\ T &= \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{4}{16} + \tfrac{5}{32} + \dots \end{aligned}\right\} \text{SUBTRACT}$$

$T = 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}} + \dots$

$T = 1 +$ (G.P WHOSE SUM TO INFINITY IS 1)

$T = 2$

$\therefore S = 2 + 2T$

$S = 2 + 2 \times 2$

$S = 6$

ALTERNATIVE USING STANDARD RESULTS FROM STATISTICS

FOR A GEOMETRIC DISTRIBUTION WITH PARAMETER p, $E(X) = \frac{1}{p}$

$\mathrm{Var}(X) = \frac{1-p}{p^2}$

NOW SUPPOSE THE PROBABILITY OF AN EVENT IS $\frac{1}{2}$, E.g. FLIPPING A FAIR COIN AND X IS THE NUMBER OF FLIPS UNTIL YOU GET HEAD

x :	1	2	3	4	5	...
$P(X=x)$:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	....
	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	.....

THUS $E(X^2) = \left(1^2 \times \frac{1}{2^1}\right) + \left(2^2 \times \frac{1}{2^2}\right) + \left(3^2 \times \frac{1}{2^3}\right) + \left(4^2 \times \frac{1}{2^4}\right) + \left(5^2 \times \frac{1}{2^5}\right) + \dots$

$= \sum_{r=1}^{\infty} \frac{r^2}{2^r}$

BUT $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$\frac{1 - \frac{1}{2}}{\frac{1}{4}} = \sum_{r=1}^{\infty} \frac{r^2}{2^r} - \left(\frac{1}{\frac{1}{2}}\right)^2$

$2 = \sum_{r=1}^{\infty} \frac{r^2}{2^r} - 4$

$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6$

Question 76 (*****)

Show clearly that

$$1 + \frac{1}{24} + \frac{1 \cdot 4}{24 \cdot 48} + \frac{1 \cdot 4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} - \dots = \frac{2}{\sqrt[3]{7}} \ .$$

proof

$S = 1 + \frac{1}{24} + \frac{1 \times 4}{24 \times 48} + \frac{1 \times 4 \times 7}{24 \times 48 \times 72} + \frac{1 \times 4 \times 7 \times 10}{24 \times 48 \times 72 \times 96} + \dots$

$S = 1 + \frac{1}{24(1)} + \frac{1 \times 4}{24^2(1 \times 2)} + \frac{1 \times 4 \times 7}{24^3(1 \times 2 \times 3)} + \frac{1 \times 4 \times 7 \times 10}{24^4(1 \times 2 \times 3 \times 4)} + \dots$

$S = 1 + \frac{3\left(\frac{1}{3}\right)}{24(1)} + \frac{3^2\left(\frac{1}{3} \times \frac{4}{3}\right)}{24^2(1 \times 2)} + \frac{3^3\left(\frac{1}{3} \times \frac{4}{3} \times \frac{7}{3}\right)}{24^3(1 \times 2 \times 3)} + \frac{3^4\left(\frac{1}{3} \times \frac{4}{3} \times \frac{7}{3} \times \frac{10}{3}\right)}{24^4(1 \times 2 \times 3 \times 4)} + \dots$

NOW FOR BINOMIAL CONVERGENT SERIES THE POWERS MUST BE DECREASING IE $n(n-1)(n-2)\dots$ ETC

THUS ADJUSTING THE SIGNS

$S = 1 + \frac{\left(-\frac{1}{3}\right)}{1}\left(-\frac{1}{8}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{1 \times 2}\left(-\frac{1}{8}\right)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{1 \times 2 \times 3}\left(-\frac{1}{8}\right)^3 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(-\frac{10}{3}\right)}{1 \times 2 \times 3 \times 4}\left(-\frac{1}{8}\right)^4 + \dots$

THIS IS THE BINOMIAL SERIES EXPANSION OF

$\left(1 - \frac{1}{8}x\right)^{-\frac{1}{3}}$ WITH $x = 1$ (NOTE THIS SERIES CONVERGES FOR $-8 < x < 8$)

$S = \left(1 - \frac{1}{8}\right)^{-\frac{1}{3}} = \left(\frac{7}{8}\right)^{-\frac{1}{3}} = \left(\frac{8}{7}\right)^{\frac{1}{3}} = \frac{2}{\sqrt[3]{7}}$

Question 77 (*****)

$$S_n = \sum_{r=1}^{n} \left(r^2 \times 2^r\right)$$

Use the standard techniques for the summation of a geometric series, to show that

$$S_n = \left(n^2 - 2n + 3\right) \times 2^{n+1} - 6 .$$

[*You may not use proof by induction in this question.*]

proof

$$\sum_{r=1}^{n} r^2 2^r = S_n = 1^2\times2^1 + 2^2\times2^2 + 3^2\times2^3 + 4^2\times2^4 + 5^2\times2^5 + \dots + n^2\times2^n$$

$$-2S_n = \quad -1^2\times2^2 - 2^2\times2^3 - 3^2\times2^4 - 4^2\times2^5 - \dots - (n-1)^2\times2^n - n^2\times2^{n+1}$$

ADD

$$-S_n = 1\times2^1 + 3\times2^2 + 5\times2^3 + 7\times2^4 + 9\times2^5 + \dots + (2n-1)\times2^n - n^2\times2^{n+1}$$

HENCE

$$+2S_n = \quad -1\times2^2 - 3\times2^3 - 5\times2^4 - 7\times2^5 - \dots - (2n-3)\times2^n - (2n-1)\times2^{n+1} + n^2\times2^{n+2}$$

ADD AGAIN

$$S_n = 1\times2 + 2\times2^2 + 2\times2^3 + 2\times2^4 + 2\times2^5 + \dots + 2\times2^n + (-n^2-2n+1)\times2^{n+1} + 2n^2\times2^{n+1}$$

$$S_n = 2 + 2\Big[\underbrace{2^2 + 2^3 + 2^4 + 2^5 + \dots + 2^n}_{\text{GP WITH } a=4,\ r=2,\ n=n-1}\Big] + (n^2-2n+1)\times2^{n+1}$$

$$S_n = 2 + 2\left(\frac{4(2^{n-1}-1)}{2-1}\right) + (n^2-2n+1)\times2^{n+1}$$

$$S_n = 2 + 8(2^{n-1}-1) + (n^2-2n+1)\times2^{n+1}$$

$$S_n = 2 + 2\times2^{n+1} - 8 + (n^2-2n+1)\times2^{n+1}$$

$$S_n = (n^2-2n+3)\times2^{n+1} - 6$$

AS REQUIRED

Question 78 (*****)

The binomial probability distribution $X \sim \mathrm{B}(n, p)$ satisfies

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

where $r = 0, 1, 2, 3, \ldots, n$ and $0 < p < 1$.

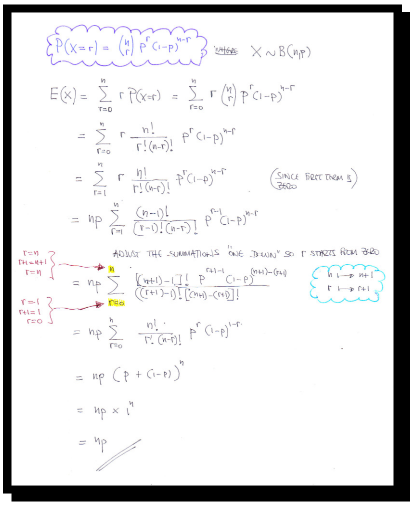
The expectation of X is defined as

$$\mathrm{E}(X) \equiv \sum_{r=0}^{n} \left[r\,\mathrm{P}(X = r) \right]$$

Show that

$$\mathrm{E}(X) = np.$$

$\boxed{\text{proof}}$



$P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ WHERE $X \sim B(n,p)$

$$E(X) = \sum_{r=0}^{n} r\,P(X=r) = \sum_{r=0}^{n} r \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$

$$= \sum_{r=1}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$ (SINCE FIRST TERM IS ZERO)

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r}$$

ADJUST THE SUMMATIONS "ONE DOWN" SO r STARTS FROM ZERO

$r=n$, $r+1=n+1$, $r=n$

$r=1$, $r+1=1$, $r=0$

$n \mapsto n+1$

$r \mapsto r+1$

$$= np \sum_{r=0}^{n} \frac{[(n+1)-1]!\, p^{r+1-1} (1-p)^{(n+1)-(r+1)}}{((r+1)-1)!\,[(n+1)-(r+1)]!}$$

$$= np \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} p^r (1-p)^{1-r}$$

$$= np\,(p + (1-p))^n$$

$$= np \times 1^n$$

$$= np$$