Question

The curve C, has gradient $\frac{1}{8}$ at the point with coordinates $\left(1,\frac{1}{2}\right)$ and further satisfies the differential relationship

$$2y^2\frac{d^2y}{dx^2}+(2y+1)(y-1)^2\frac{dy}{dx}=0, \quad y \neq 0.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y=\frac{\sqrt{x}}{1+\sqrt{x}}}$$

$$\boxed{2y^2\frac{d^2y}{dx^2}+(2y+1)(y-1)^2\frac{dy}{dx}=0 \qquad x=1,\ y=\tfrac{1}{2},\ \frac{dy}{dx}=\tfrac{1}{8}}$$

$$\Rightarrow 2y^2\frac{d^2y}{dx^2}=-(2y+1)(y-1)^2\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(2y+1)(y^2-2y+1)}{2y^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{\begin{array}{c}2y^3-4y^2+2y\\ y^2-2y+1\end{array}}{2y^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{2y^3-3y^2+1}{2y^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(-y+\frac{3}{2}-\frac{1}{2}y^{-2}\right)\frac{dy}{dx}$$

- INTEGRATE BOTH SIDES WITH RESPECT TO x SUBJECT TO $y=\frac{1}{2}$ $dy/dx=\frac{1}{8}$

$$\Rightarrow \int_{\frac{1}{8}}^{y'}\frac{d^2y}{dx^2}\,dx=\int_{\frac{1}{2}}^{y}\left(-y+\frac{3}{2}-\frac{1}{2}y^{-2}\right)\cancel{\frac{dy}{dx}}\,\cancel{dx}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\frac{1}{8}}^{y'}=\left[-\frac{1}{2}y^2+\frac{3}{2}y+\frac{1}{2}y^{-1}\right]_{\frac{1}{2}}^{y}$$

$$\Rightarrow \frac{dy}{dx}-\frac{1}{8}=\left(-\frac{1}{2}y^2+\frac{3}{2}y+\frac{1}{2y}\right)-\left(-\frac{1}{8}+\frac{3}{4}+1\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2}y^2+\frac{3}{2}y-\frac{3}{2}+\frac{1}{2y}$$

TIDY UP FIRST

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2y}\left[y^3-3y^2+3y-1\right]$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(y-1)^3}{2y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(1-y)^3}{2y}$$

- SEPARATE VARIABLES

$$\Rightarrow \frac{2y}{(1-y)^3}\,dy=1\,dx$$

- INTEGRATE SUBJECT TO THE CONDITIONS $x=1, y=\frac{1}{2}$

$$\Rightarrow \int_{\frac{1}{2}}^{y}\frac{2y}{(1-y)^3}\,dy=\int_{1}^{x}1\,dx$$

- SUBSTITUTION (OR PARTIAL FRACTIONS)

$u=1-y$

$y=1-u$

$dy=-du$

$y=\frac{1}{2}\longmapsto u=\frac{1}{2}$

$y=y\longmapsto u=1-y$

$$\Rightarrow \int_{\frac{1}{2}}^{1-y}\frac{2-2u}{u^3}(-du)=\int_{1}^{x}1\,dx$$

$$\Rightarrow 2\int_{\frac{1}{2}}^{1-y}-u^{-3}+u^{-2}\,du=\int_{1}^{x}1\,dx$$

$$\Rightarrow 2\left[\frac{1}{2}u^{-2}-u^{-1}\right]_{\frac{1}{2}}^{1-y}=\left[x\right]_{1}^{x}$$

$$\Rightarrow \left[\frac{1}{u^2}-\frac{2}{u}\right]_{\frac{1}{2}}^{1-y}=x-1$$

$$\Rightarrow \left[\frac{1-2u}{u^2}\right]_{\frac{1}{2}}^{1-y}=x-1$$

$$\Rightarrow \frac{1-2(1-y)}{(1-y)^2}-0=x-1$$

$$\Rightarrow \frac{2y-1}{(y-1)^2}=x-1$$

$$\Rightarrow 2y-1=(x-1)(y^2-2y+1)$$

$$\Rightarrow 2y-1=(x-1)y^2-2(x-1)y+(x-1)$$

$$\Rightarrow \cancel{2y}\cancel{-1}=(x-1)y^2-2xy\cancel{+2y}+x\cancel{-1}$$

$$\Rightarrow (x-1)y^2-2xy+x=0 \qquad \boxed{\left[y(\sqrt{x}-1)-\sqrt{x}\right]\left[y[\sqrt{x}+1]-\sqrt{x}\right]=0}$$

FACTORIZE OR COMPLETE THE SQUARE

$$\Rightarrow y^2-\frac{2x}{x-1}y+\frac{x}{x-1}=0$$

$$\Rightarrow \left[y-\frac{x}{x-1}\right]^2-\frac{x^2}{(x-1)^2}+\frac{x}{x-1}=0$$

$$\Rightarrow \left[y-\frac{x}{x-1}\right]^2+\frac{-x^2+x(x-1)}{(x-1)^2}=0$$

$$\Rightarrow \left[y-\frac{x}{x-1}\right]^2+\frac{-x}{(x-1)^2}=0$$

$$\Rightarrow \left[y-\frac{x}{x-1}\right]^2=\frac{x}{x-1}$$

$$\Rightarrow y-\frac{x}{x-1}=\frac{\pm\sqrt{x}}{x-1}$$

$$\Rightarrow y=\frac{x\pm\sqrt{x}}{x-1}$$

$$\Rightarrow y=\frac{\sqrt{x}(\sqrt{x}\pm 1)}{(\sqrt{x})^2-1}$$

$$\Rightarrow y=\frac{\sqrt{x}\left[\sqrt{x}\pm 1\right]}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$\Rightarrow y=\begin{cases}\cancel{\dfrac{\sqrt{x}}{\sqrt{x}-1}} & \text{NOT DEFINED AT } x=1\\[2mm] \dfrac{\sqrt{x}}{\sqrt{x}+1}\end{cases}$$

$$\Rightarrow y=\frac{\sqrt{x}}{\sqrt{x}+1}$$

ALTERNATIVE REARRANGEMENT.

$$\Rightarrow \frac{2y-1}{(y-1)^2}=x-1$$

$$\Rightarrow x=\frac{2y-1}{(y-1)^2}+1$$

$$\Rightarrow x=\frac{(2y-1)+(y-1)^2}{(y-1)^2}$$

$$\Rightarrow x=\frac{2y-1+y^2-2y+1}{(y-1)^2}$$

$$\Rightarrow x=\frac{y^2}{(y-1)^2}$$

$$\Rightarrow \frac{y}{y-1}=\begin{cases}\sqrt{x}\\ -\sqrt{x}\end{cases}$$

ORIGINAL IS SATISFIED BY $x=1$, $y=\frac{1}{2}$

HENCE

$$\frac{y}{y-1}=-\sqrt{x}$$

$$y=-y\sqrt{x}+\sqrt{x}$$

$$y+y\sqrt{x}=\sqrt{x}$$

$$y(1+\sqrt{x})=\sqrt{x}$$

$$y=\frac{\sqrt{x}}{1+\sqrt{x}}$$

AS BEFORE