

**Question** (\*\*\*)

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right) + \arccos\left(10^{-\frac{1}{2}}\right) = \frac{3\pi}{4}.$$

proof

Handwritten proof for the identity:

$$\arccos\left(\frac{1}{\sqrt{5}}\right) + \arccos\left(\frac{1}{\sqrt{10}}\right) = \frac{3\pi}{4}$$

Let  $\theta = \arccos\left(\frac{1}{\sqrt{5}}\right)$  and  $\phi = \arccos\left(\frac{1}{\sqrt{10}}\right)$ .

From the definition of arccos, we have:

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\phi = \frac{1}{\sqrt{10}}$$

Geometric diagrams show right-angled triangles with hypotenuses 1, adjacent sides 1/√5 and 1/√10, and angles θ and φ respectively.

Using the cosine addition formula:

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} = \cos\alpha$$

$$\frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}} = \cos\alpha$$

$$\cos\alpha = -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}}$$

$$\alpha = \frac{3\pi}{4}$$

Therefore,  $\arccos\left(\frac{1}{\sqrt{5}}\right) + \arccos\left(\frac{1}{\sqrt{10}}\right) = \frac{3\pi}{4}$ .

Alternative method - Tangents

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$$

$$\tan\theta = \frac{2}{1} = 2 \quad \text{and} \quad \tan\phi = \frac{3}{1} = 3$$

$$\tan(\theta + \phi) = \frac{2 + 3}{1 - 2 \times 3} = -\frac{5}{5} = -1$$

$$\theta + \phi = -\frac{\pi}{4}, \frac{3\pi}{4}, \dots$$

Since  $0 < \arccos\left(\frac{1}{\sqrt{5}}\right) < \frac{\pi}{2}$  and  $0 < \arccos\left(\frac{1}{\sqrt{10}}\right) < \frac{\pi}{2}$ , we have:

$$0 < \arccos\left(\frac{1}{\sqrt{5}}\right) + \arccos\left(\frac{1}{\sqrt{10}}\right) < \pi$$

Therefore,  $\arccos\left(\frac{1}{\sqrt{5}}\right) + \arccos\left(\frac{1}{\sqrt{10}}\right) = \frac{3\pi}{4}$ .