Question

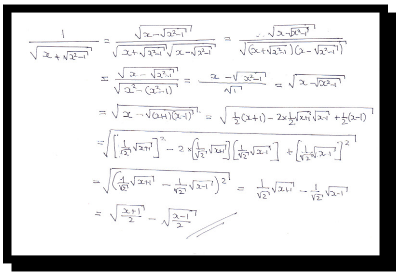
$$h(x) \equiv \frac{1}{\sqrt{x+\sqrt{x^2-1}}}, \ x \in \mathbb{R}, \ x \geq 1.$$

Show that $h(x)$ can be expressed in the form

$$\sqrt{f(x)} - \sqrt{g(x)},$$

where $f(x)$ and $g(x)$ are linear functions to be found.

$$\boxed{h(x) = \sqrt{\frac{x+1}{2}} - \sqrt{\frac{x-1}{2}}}$$



$$\frac{1}{\sqrt{x+\sqrt{x^2-1}}} = \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{x+\sqrt{x^2-1}}\sqrt{x-\sqrt{x^2-1}}} = \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}}$$

$$= \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{x^2-(x^2-1)}} = \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{1}} = \sqrt{x-\sqrt{x^2-1}}$$

$$= \sqrt{x-\sqrt{(x+1)(x-1)}} = \sqrt{\tfrac{1}{2}(x+1) - 2\times\tfrac{1}{2}\sqrt{x+1}\sqrt{x-1} + \tfrac{1}{2}(x-1)}$$

$$= \sqrt{\left[\tfrac{1}{\sqrt{2}}\sqrt{x+1}\right]^2 - 2\times\left[\tfrac{1}{\sqrt{2}}\sqrt{x+1}\right]\left[\tfrac{1}{\sqrt{2}}\sqrt{x-1}\right] + \left[\tfrac{1}{\sqrt{2}}\sqrt{x-1}\right]^2}$$

$$= \sqrt{\left(\tfrac{1}{\sqrt{2}}\sqrt{x+1} - \tfrac{1}{\sqrt{2}}\sqrt{x-1}\right)^2} = \frac{1}{\sqrt{2}}\sqrt{x+1} - \frac{1}{\sqrt{2}}\sqrt{x-1}$$

$$= \sqrt{\frac{x+1}{2}} - \sqrt{\frac{x-1}{2}}$$