

Question 9

The product operator \prod , is defined as

$$\prod_{i=1}^k [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \dots \times u_{k-1} \times u_k.$$

Evaluate, showing a clear method

$$\prod_{r=2}^{\infty} \left[\frac{r^3 - 1}{r^3 + 1} \right].$$

$$\boxed{\frac{2}{3}}$$

Handwritten solution for the infinite product problem:

$$\begin{aligned} \prod_{r=2}^{\infty} \left[\frac{r^3 - 1}{r^3 + 1} \right] &= \lim_{k \rightarrow \infty} \left[\prod_{r=2}^k \left(\frac{(r-1)(r^2+r+1)}{(r+1)(r^2-r+1)} \right) \right] \\ &= \lim_{k \rightarrow \infty} \left[\frac{(1)(3)}{2(2)} \times \frac{2(3)}{3(3)} \times \frac{3(4)}{4(4)} \times \frac{4(5)}{5(5)} \times \dots \times \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} \right] \\ &= \lim_{k \rightarrow \infty} \left[\frac{(1)(3)(4)(5) \dots (k-1)(k^2+k+1)}{2(2)(3)(3)(4)(4) \dots (k+1)(k^2-k+1)} \right] \\ &= \lim_{k \rightarrow \infty} \left[\frac{(1)(3)}{2(2)} \times \frac{k^2+k+1}{k^2-k+1} \right] = \lim_{k \rightarrow \infty} \left[\frac{2}{3} \times \frac{k^2+k+1}{k^2-k+1} \right] \\ &= \frac{2}{3} \end{aligned}$$