

Question non calculator

By using trigonometric identities, show that

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \frac{1}{8}(16-3\pi).$$

proof

The image shows a handwritten proof of the integral. The steps are as follows:

$$\begin{aligned} \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\sin^2 x \cos^2 x} + \frac{\cos^4 x}{\sin^2 x \cos^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(1-\cos^2 x)^2}{\cos^2 x} + \frac{(1-\sin^2 x)^2}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1-2\cos^2 x + \cos^4 x}{\cos^2 x} + \frac{1-2\sin^2 x + \sin^4 x}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x - 2 + \frac{\cos^2 x}{\cos^2 x} + \csc^2 x - 2 + \frac{\sin^2 x}{\sin^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} -3 + \sec^2 x + \csc^2 x dx \\ &= \left[-3x + \tan x - \cot x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = \left[-3x + \tan x - \frac{1}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[-3x + \frac{\tan^2 x - 1}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = \left[-3x - 2 \left(\frac{1 - \tan^2 x}{2 \tan x} \right) \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[-3x - \frac{1 - \tan^2 x}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = \left(-\frac{3\pi}{4} - 0 \right) - \left(-\frac{3\pi}{8} - 2 \right) = 2 - \frac{3\pi}{8} \\ &= \frac{1}{8}(16-3\pi) \end{aligned}$$