

## Question

By using trigonometric identities, show that

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \frac{3\pi}{16}.$$

proof

The image shows a handwritten mathematical proof for the integral  $\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \frac{3\pi}{16}$ . The proof is written in black ink on a white background and is enclosed in a black rectangular border. It starts with the integral  $\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx$  and uses the identity  $(\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x$  to rewrite the integrand as  $(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$ . This leads to  $\int_0^{\frac{\pi}{4}} (1 - 2\sin^2 x \cos^2 x) \, dx$ . The next step is to use the double-angle identity  $\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x$ , resulting in  $\int_0^{\frac{\pi}{4}} (1 - \frac{1}{2} \sin^2 2x) \, dx$ . This is then integrated to  $\frac{x}{1} - \frac{1}{4} \sin 2x \cos 2x$  evaluated from 0 to  $\frac{\pi}{4}$ . The final result is  $\frac{3\pi}{16}$ . A note at the bottom states:  $\cos^2 x \equiv \frac{1}{2} + \frac{1}{2} \cos 2x$  and  $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2} \cos 2x$ .

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx &= \int_0^{\frac{\pi}{4}} ((\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x) - 2\sin^2 x \cos^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x)^2 - \frac{1}{2}(4\sin^2 x \cos^2 x) \, dx = \int_0^{\frac{\pi}{4}} (1 - \frac{1}{2} \sin^2 2x) \, dx \\ &= \int_0^{\frac{\pi}{4}} (1 - \frac{1}{2} \sin^2 2x) \, dx = \int_0^{\frac{\pi}{4}} (1 - \frac{1}{4} (1 - \cos 4x)) \, dx \\ &= \int_0^{\frac{\pi}{4}} (\frac{3}{4} + \frac{1}{4} \cos 4x) \, dx = \left[ \frac{3}{4}x + \frac{1}{16} \sin 4x \right]_0^{\frac{\pi}{4}} \\ &= \left( \frac{3\pi}{16} - 0 \right) - (0 - 0) = \frac{3\pi}{16} \end{aligned}$$

ALTERNATIVE VARIATION

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx &= \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right)^2 + \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} \left( \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x + \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} + \frac{1}{2} \cos^2 2x \right) \, dx = \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} + \frac{1}{4} (1 + \cos 4x) \right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \left( \frac{3}{4} + \frac{1}{4} \cos 4x \right) \, dx = \dots = \frac{3\pi}{16} \text{ as above} \end{aligned}$$

NOTE  $\cos^2 x \equiv \frac{1}{2} + \frac{1}{2} \cos 2x$   
 $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2} \cos 2x$