Question

By using trigonometric identities, show that

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \ dx = \frac{3\pi}{16} .$$

$\boxed{\text{proof}}$

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} \left[(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x \right] - 2\sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x)^2 - \tfrac{1}{2}(4\sin^2 x \cos^2 x) \, dx = \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}(\sin 2x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}\sin^2 2x \, dx = \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{3}{4} + \tfrac{3}{4}\cos 4x \, dx = \left[\tfrac{3}{4}x + \tfrac{3}{16}\sin 4x \right]_0^{\frac{\pi}{4}}$$

$$= \left(\tfrac{3\pi}{16} - 0\right) - (0 - 0) = \tfrac{3\pi}{16}$$

ALTERNATIVE VARIATION

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)^2 dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{4} \cancel{- \tfrac{1}{2}\cos 2x} + \tfrac{1}{4}\cos^2 2x + \tfrac{1}{4} \cancel{+ \tfrac{1}{2}\cos 2x} + \tfrac{1}{4}\cos^2 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{2} + \tfrac{1}{2}\cos^2 2x \, dx = \int \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{3}{4} + \tfrac{1}{4}\cos 4x \, dx = \dots = \tfrac{3\pi}{16} \text{ AS ABOVE}$$

NOTE $\cos^2\theta \equiv \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta$

$\sin^2\theta \equiv \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$