Question

Use appropriate methods to find, in terms of k, a simplified expression for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\tan^2 x}\ dx, \quad |k| \neq 1.$$

$$\boxed{\frac{\pi}{2(k+1)}}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{k^2\tan^2 x+1}\ dx = \ldots \text{ BY SUBSTITUTION} \ldots$$

$$u = k\tan x$$
$$\frac{du}{dx} = k\sec^2 x$$
$$dx = \frac{du}{k\sec^2 x}$$
$$dx = \frac{du}{k(\tan^2 x+1)}$$
$$dx = \frac{du}{k\left(\frac{u^2}{k^2}+1\right)}$$
$$x=0 \longmapsto u=0$$
$$x=\frac{\pi}{2} \longmapsto u=\infty$$

$$= \int_0^{\infty} \frac{1}{u^2+1}\ \frac{du}{k\left(\frac{u^2}{k^2}+1\right)} = \int_0^{\infty} \frac{1}{u^2+1}\ \frac{k}{u^2+k^2}\ du$$

PARTIAL FRACTIONS

$$\frac{k}{(u^2+1)(u^2+k^2)} \equiv \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2+k^2}$$
$$k \equiv (Au+B)(u^2+k^2) + (Cu+D)(u^2+1)$$
$$k \equiv Au^3 + Bu^2 + Ak^2u + Bk^2$$
$$Cu^3 + Du^2 + Cu + D$$

- $A+C=0$
- $B+D=0$
- $Ak^2+C=0$
- $Bk^2+D=k$

Thus $\boxed{D = k - Bk^2}$ & $\boxed{C = -Ak^2}$

$$B + (k - Bk^2) = 0 \qquad A + (Ak^2) = 0$$
$$B - Bk^2 = -k \qquad A(1+k^2) = 0$$
$$B(1-k^2) = -k \qquad \boxed{A=0}$$
$$\boxed{B = \frac{k}{k^2-1}} \qquad \boxed{C=0}$$
$$\boxed{D = -\frac{k}{k^2-1}}$$

$$\ldots = \int_0^{\infty} \frac{\frac{k}{k^2-1}}{u^2+1} - \frac{\frac{k}{k^2-1}}{u^2+k^2}\ du = \frac{k}{k^2-1}\int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2}\ du$$

$$= \frac{k}{k^2-1}\left[\arctan u - \frac{1}{k}\arctan\frac{u}{k}\right]_0^{\infty} = \frac{k}{k^2-1}\left[\left(\frac{\pi}{2} - \frac{1}{k}\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{k}{k^2-1} \times \frac{\pi}{2}\left(1 - \frac{1}{k}\right) = \frac{\pi}{2}\ \frac{k}{k^2-1} \times \frac{k-1}{k} = \frac{\pi k(k-1)}{2k(k+1)(k-1)}$$

$$= \frac{\pi}{2(k+1)}$$