Question

$$I=\int_{0}^{1}\left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\left(\frac{1}{x+r}\right)\right]dx .$$

Show by a detailed method that

$$I=a\times b!,$$

where a and b are positive integers to be found.

The product operator $\prod$, ***is defined as***

$$\prod_{i=1}^{k}\left[u_i\right] \;=\; u_1\times u_2\times u_3\times u_4\times\ldots\times u_{k-1}\times u_k .$$

$$\boxed{a=b=10}$$

$$\int_0^1\left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right]dx$$

- LET $u=\prod_{r=1}^{10}(x+r) = (x+1)(x+2)(x+3)\ldots(x+10)$

$$\ln u = \ln\left[(x+1)(x+2)(x+3)\ldots(x+10)\right]$$

$$\ln u = \ln(x+1)+\ln(x+2)+\ln(x+3)+\ldots+\ln(x+10)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+10}$$

$$\frac{du}{dx}\times\frac{1}{u} = \sum_{r=1}^{10}\frac{1}{x+r}$$

$$\frac{du}{dx} = u\sum_{r=1}^{10}\frac{1}{x+r}$$

$$\frac{du}{dx} = \prod_{r=1}^{10}(x+r)\sum_{r=1}^{10}\frac{1}{x+r}$$

- $\frac{dv}{dx} = \sum_{r=1}^{\infty}\frac{1}{x+r} = \frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\ldots+\frac{1}{x+10}$

$$v = \ln(x+1)+\ln(x+2)+\ln(x+3)+\ldots+\ln(x+10)$$

$$v = \ln\left[(x+1)(x+2)(x+3)\ldots(x+10)\right]$$

$$v = \ln\left[\prod_{r=1}^{10}(x+r)\right]$$

- BY PARTS

$$\int_0^1\left[\underbrace{\prod_{r=1}^{10}(x+r)}_{u}\right]\left[\underbrace{\sum_{r=1}^{\infty}\frac{1}{(x+r)}}_{\frac{dv}{dx}}\right]dx = \left[\overbrace{\prod_{r=1}^{10}(x+r)}^{u}\,\overbrace{\ln\prod_{r=1}^{10}(x+r)}^{v}\right]_0^1$$

$$-\int_0^1\underbrace{\prod_{r=1}^{10}(x+r)\sum_{r=1}^{10}\frac{1}{x+r}}_{\frac{du}{dx}}\,\underbrace{\ln\left[\prod_{r=1}^{10}(x+r)\right]}_{v}\,dx$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\int_0^1\frac{du}{dx}v\,dx$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\int_{x=0}^{x=1}\frac{du}{dx}\ln(u)\,dx$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\int_{x=0}^{x=1}\ln u\,du$$

EARLIER

$$v=\ln\left[\prod_{r=1}^{10}(x+r)\right]$$

$$u=\prod_{r=1}^{10}(x+r)$$

$$\therefore v=\ln(u)$$

STANDARD RESULTS OR BY PARTS

$$\int \ln y\,dy = y\ln y - y + C$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\left\{\Big[u\ln u - u\Big]_{x=0}^{x=1}\right\}$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\Big[u\ln u\Big]_{x=0}^{x=1}+\Big[u\Big]_{x=0}^{x=1}$$

$$=\cancel{\Big[uv\Big]_{x=0}^{x=1}}-\cancel{\Big[uv\Big]_{x=0}^{x=1}}+\left[\prod_{r=1}^{10}(x+r)\right]_0^1$$

$$=\prod_{r=1}^{10}(1+r)-\prod_{r=1}^{10}r = 11!-10!$$

$$= 11\times 10! - 10!$$

$$= 10\times 10!$$