1990 STEP I number 4

4. $S_n = \frac{n}{2}(a_1 + a_n)$ and $a_n = a_1 + (n-1)d$ where $d = a_2 - a_1$

hence, $a_n = a_1 + (n-1)(a_2 - a_1) = (2-n)a_1 + (n-1)a_2$

and so $S_n = \frac{n}{2}\left[(3-n)a_1 + (n-1)a_2\right]$

Consider the first r terms of the sequence $b_2 - b_1, b_3 - b_2, ..., b_{n+1} - b_n$

then $S_r = b_{r+1} - b_1$ and by the previous result, $S_r = \frac{r}{2}\left[(3-r)(b_2 - b_1) + (r-1)(b_3 - b_2)\right]$

hence, $n_{r+1} = b_1 + \frac{r}{2}\left[(3-r)(b_2 - b_1) + (r-1)(b_3 - b_2)\right]$

$= b_1\left[1 + \frac{r}{2}(r-3)\right] + b_2\left[\frac{r}{2}(3-r) - \frac{r}{2}(r-1)\right] + b_3\frac{r}{2}(r-1)$

$= b_1\left(\frac{1}{2}(r^2 - 3r + 2)\right) + b_2(r(2-r)) + b_3\left(\frac{r}{2}(r-1)\right)$

so $b_r = b_1\left(\frac{1}{2}(r^2 - 5r + 6)\right) - b_2(r^2 - 4r + 3) + b_3\left(\frac{1}{2}(r^2 - 3r + 2)\right)$

hence, $b_1 + b_2 + b_3 + ... + b_n = b_1 \sum_{r=1}^{n} \frac{1}{2}(r^2 - 5r + 6) - b_2 \sum_{r=1}^{n} (r^2 - 4r + 3) + b_3 \sum_{r=1}^{n} \frac{1}{2}(r^2 - 3r + 2)$

$= b_1\left[\frac{n}{12}(n+1)(2n+1) - \frac{5}{4}n(n+1) + 3n\right] - b_2\left[\frac{n}{6}(n+1)(2n+1) - 2n(n+1) + 3n\right]$

$+ b_3\left[\frac{n}{12}(n+1)(2n+1) - \frac{3}{4}n(n+1) + n\right]$

$= \frac{1}{12}(2n^3 - 12n^2 + 22n)b_1 - \frac{1}{6}(2n^3 - 9n^2 + 7n)b_2 + \frac{1}{12}(2n^3 - 6n^2 + 4n)b_3$

$b_3 = 2b_2 - b_1 + k,\ b_4 = 2b_3 - b_2 + k = 3b_2 - 2b_1 + 3k,$

$b_5 = 2b_4 - b_3 + k = 6b_2 - 4b_1 + 6k - 2b_2 + b_1 - k + k = 4b_2 - 3b_1 + 6k$

$b_6 = 2b_5 - b_4 + k = 8b_2 - 6b_1 + 12k - 3b_2 + 2b_1 - 3k + k = 5b_2 - 4b_1 + 10k$

$c_2 = c_1 + d,\ c_3 = c_2 + 2d = c_1 + 3d,\ c_4 = c_3 + 6d = c_1 + 9d$ and $c_1 = b_2,\ b_1 = 1$

hence, $c_2 = b_4 \Rightarrow b_2 + d = 3b_2 - 2 + 3k$

$c_3 = b_6 \Rightarrow b_2 + 3d = 5b_2 - 4 + 7k$ and $c_4 = 220 \Rightarrow b_2 + 9d = 220$

so $b_2 = 220 - 9d \Rightarrow$ and $c_2 = b_5 \Rightarrow 220 - 8d = 660 - 27d - 2 + 3k \Rightarrow 19d - 3k = 438$

and $c_3 = b_6 \Rightarrow 220 - 6d = 1100 - 45d - 4 + 8k \Rightarrow 39d - 10k = 876 \Rightarrow d = 4k$

hence $73k = 438 \Rightarrow k = 6,\ d = 24,\ b_2 = 220 - 216 = 4,\ b_3 = 13$

so using earlier result,

$b_1 + b_2 + ... + b_n = \frac{1}{12}(2n^3 - 12n^2 + 22n) - \frac{4}{6}(2n^3 - 9n^2 + 7n) + \frac{13}{12}(2n^3 - 6n^2 + 4n)$

$= \frac{1}{12}(12n^3 - 18n^2 + 18n) = \frac{1}{2}n(2n^2 - 3n + 3)$

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