11. By conservation of angular momentum,

$$mk^2\Omega + ma^2\Omega = mk^2\omega + mr^2\omega \Rightarrow \omega = \frac{k^2+a^2}{k^2+r^2}\Omega$$

By conservation of kinetic energy, since no external work is done

$$\frac{1}{2}mk^2\Omega^2 + \frac{1}{2}ma^2\Omega^2 + \frac{1}{2}mV^2 = \frac{1}{2}mk^2\omega^2 + \frac{1}{2}mr^2\omega^2 + \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2$$

$$\Rightarrow k^2\Omega^2 + a^2\Omega^2 + V^2 = k^2\omega^2 + r^2\omega^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2$$

$$\Rightarrow \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = k^2\Omega^2 + a^2\Omega^2 + V^2 - k^2\omega^2 - r^2\omega^2 = \left(k^2+a^2\right)\Omega^2 + V^2 - \left(k^2+r^2\right)\omega^2$$

$$= \left(k^2+a^2\right)\Omega^2 + \frac{\Omega^2a^2(k^2+a^2)}{k^2} - \left(k^2+r^2\right)\left(\frac{k^2+a^2}{k^2+r^2}\right)^2\Omega^2$$

$$= \left(\left(k^2+a^2\right) - \frac{(k^2+a^2)^2}{k^2+r^2} + \frac{a^2(k^2+a^2)}{k^2}\right)\Omega^2 = \frac{(k^2+a^2)(k^2(k^2+r^2)-k^2(k^2+a^2)+a^2(k^2+r^2)}{k^2(k^2+r^2)}\Omega^2$$

$$= \frac{(k^2+a^2)(k^2r^2+a^2r^2)}{k^2(k^2+r^2)}\Omega^2 = \frac{(k^2+a^2)^2r^2\Omega^2}{k^2(k^2+r^2)} \text{ so } \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{r(k^2+a^2)}{k\sqrt{k^2+r^2}}\Omega$$

but r is decreasing so we require the negative root i.e. $\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{r(k^2+a^2)}{k\sqrt{k^2+r^2}}\Omega$

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = k\frac{\mathrm{d}r}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{k}{\omega}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k(k^2+r^2)}{(k^2+a^2)\Omega}\times -\frac{r(k^2+a^2)}{k\sqrt{k^2+r^2}}\Omega = -r\sqrt{k^2+r^2} \text{ as required}$$

putting $u = \frac{1}{r}$ so $r = \frac{1}{u} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}u} = -\frac{1}{u^2}$ so $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}$

so $-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{r}{k}\sqrt{k^2+r^2} = -\frac{1}{uk}\sqrt{k^2+\frac{1}{u^2}} = -\frac{1}{u^2k}\sqrt{u^2k^2+1}$

$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{\sqrt{u^2k^2+1}}{k} \Rightarrow \int\frac{\mathrm{d}u}{\sqrt{u^2k^2+1}} = \int\frac{1}{k}\mathrm{d}\theta \text{ so } \frac{\theta}{k} = \frac{1}{k}\sinh^{-1}(uk) + C \Rightarrow ku = \sinh(\theta - C)$$

i.e. $k = r\sinh(\theta - C)$ and $\theta = 0$ when $r = a \Rightarrow \sinh(-C) = \frac{k}{a} = -\sinh C$

hence, $r\sinh(\theta + \alpha) = k$ where $\sinh\alpha = \frac{k}{a}$

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