2. Let radii of body and head be x and y respectively, then volume of head is $\frac{4}{3}\pi x^3$ and volume of body is $\frac{4}{3}\pi y^3$. Surface areas are $4\pi x^2$ and $4\pi y^2$ so rate of melting is $4\pi x^2\frac{\mathrm{d}x}{\mathrm{d}t} = -4k\pi x^2$ and $4\pi y^2\frac{\mathrm{d}y}{\mathrm{d}t} = -4k\pi y^2$ respectively, hence $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = -k \Rightarrow x = -kt + 2R$ and $y = -kt + 3R$

Total height is $h = 2x + 2y = 10R - 4kt$ so height is halved when $4kt = 5R \Rightarrow t = \frac{5R}{4k}$

radii of body and head are then $\frac{3}{4}R$ and $\frac{7}{4}R$ so volume of Frosty is $\frac{4}{3}\pi\left(\frac{27}{64} + \frac{343}{64}\right)R^3 = \frac{370}{48}\pi R^3$ and surface area is $4\pi\left(\frac{9}{16} + \frac{49}{16}\right)R^2 = \frac{29}{2}\pi R^2$

Ratio of volume to initial volume is thus $\frac{\frac{370}{48}\pi R^3}{\frac{4}{3}\pi \times 35R^3} = \frac{370}{48} \times \frac{23}{140} = \frac{37}{224}$

His head disappears when $x = 0 \Rightarrow t = \frac{2R}{k}$

Surface area $A = 4\pi\left((2r - kt)^2 + (3R - kt)^2\right) = 4\pi\left(13R^2 - 10kRt + 2k^2t^2\right)$

Volume $V = \frac{4}{3}\pi\left((2r - kt)^3 + (3R - kt)^3\right) = \frac{4}{3}\pi\left(35R^3 - 39kR^2t + 15k^2Rt^2 - 2k^3t^3\right)$

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4}{3}\pi\left(-39kR^2 + 30k^2Rt - 6k^3t^2\right)$ and $\frac{\mathrm{d}S}{\mathrm{d}t} = 4\pi\left(4k^2t - 10kR\right)$

so $\frac{\mathrm{d}V}{\mathrm{d}S} = \frac{13R^2 - 10kRt + 2k^2t^2}{10R - 4kt} \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}S^2} = \frac{(10R - 4kt)(4k^2t - 10kR) + 4k(13R^2 - 10kRt + 2k^2t^2)}{(10R - 4kt)^2} = 0$ if

$-8k^3t^2 + 40k^2Rt - 48kR^2 = 0 \Rightarrow k^2t^2 - 5kRt + 6R^2 = 0 \Rightarrow (kt - 3R()(kt - 2R) = 0$

i.e. $t = \frac{3R}{k}$ or $\frac{2R}{k}$ but $t = \frac{3R}{k}$ is outside our possible range of values.

$t = \frac{2R}{k} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}S} = \frac{13R^2 - 20R^2 + 8R^2}{(2R)^2} = \frac{1}{4}$ so maximum value of $\frac{\mathrm{d}V}{\mathrm{d}S}$ is $\frac{1}{4}$ when $t = \frac{2R}{k}$

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