7. For first year $\frac{dV}{dt} = aV_0\left(1+\frac{V}{V_0}\right)^2 = \frac{a(V_0+V)^2}{V_0} \Rightarrow \int \frac{dV}{(V_0+V)^2} = \int \frac{a}{V_0}dt$

so $-\frac{1}{V_0+V} = \frac{a}{V_0}t - \frac{1}{V_0(1+k_0)}$ since $V = k_0V_0$ at $t = 0$

i.e. $\frac{1}{V_0+V} = \frac{1}{(1+k)V_0} - \frac{a}{V_0}t = \frac{V_0-a(1+k_0)V_0t}{V_0^2(1+k_0)} \Rightarrow V = \frac{V_0(1+k_0)}{1-a(1+k_0)t} - V_0$

clearly, since the starting amount in the treasury is k_0V_0 then we must have $k_0 \geq 0$

At end of first year $t = 1 \Rightarrow V = \frac{V_0(1+k_0)}{1-a(1+k_0)} - V_0 = \frac{k_0+a(1+k_0)}{1-a(1+k_0)}V_0$ which is $> k_0V_0$ if

$a(1+k_0) < 1 \Rightarrow ak_0 < 1-a \Rightarrow k_0 < a^{-1}-1$ so this condition is required for the economy to have grown

For second year $\frac{dV}{dt} = -bV_0\left(1 = \frac{V}{V_0}\right)^2 = -\frac{b(V_0+V)^2}{V_0} \Rightarrow \int \frac{-dv}{(V_0+V)^2} = \int \frac{b}{V_0}dt$

so $\frac{1}{V_0+V} = \frac{bt}{V_0} + \frac{1}{V_0+k_1V_0} = \frac{1+b(1+k_1)t}{V_0(1+k_1)} \Rightarrow V = \frac{V_0(1+k_1)}{1+b(1=k_1)t} - V_0$

aso at end of second year $V = \frac{k_1(1-b)-b}{1+b(1+k_1)}V_0 > 0$ if $k_1 - bk_1 > b$

i.e. required condition is $k_1 > \frac{b}{1-b}$

In third year $V = \frac{V_0(1+k_2)}{1-a(1+k_2)t} - V_0$ with $k_2 = \frac{k_0+a(1+k_0)}{1-(1+k_0)}$

unlimited growth occurs in third year if $1-a(1+k_2) = 0$, i.e. $a + ak_2 \geq 1 \Rightarrow k_2 \geq \frac{1-a}{a}$

hence, $\frac{k_0+a(1+k_0)}{1-(1+k_0)} \geq \frac{1-a}{a} \Rightarrow ak_0 + a^2(1+k_0) \geq 1-a-a(1+k_0)+a^2(1-k_0)$

$\Rightarrow ak_0 \geq 1-2a-ak_0 \Rightarrow k_0 \geq \frac{1-2a}{2a}$

Unlimited growth does not occur in first year if $1-a(1+k_0) > 0 \Rightarrow k_0 < \frac{1-a}{a}$

so required conditions are $\frac{1-2a}{2a} \leq k_0 < \frac{1-a}{a}$
