Let $\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. Let $\mathbf{M}$ be such that $\mathbf{Ma} = \mathbf{p}$ and $\mathbf{Mb} = \mathbf{q}$.
Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and likewise for $\mathbf{b}$, $\mathbf{c}$, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.
Multiplying out, we obtain a system of four equations:

$$\begin{cases} M_{11}a_1 + M_{12}a_2 &= p_1 \\ M_{11}b_1 + M_{12}b_2 &= q_1 \\ M_{21}a_1 + M_{22}a_2 &= p_2 \\ M_{21}b_1 + M_{22}b_2 &= q_2 \end{cases}$$

We may express this system in matrix form:

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ b_1 & b_2 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} M_{11} \\ M_{12} \\ M_{21} \\ M_{22} \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix}$$

Let the 4×4 matrix be denoted by $\mathbf{T}$.

$$\det \mathbf{T} = a_1 \begin{vmatrix} b_2 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & b_1 & b_2 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & b_1 & b_2 \end{vmatrix} = a_1 b_2 (a_1 b_2 - a_2 b_1) - a_2 b_1 (a_1 b_2 - a_2 b_1)$$

$$\det \mathbf{T} = (a_1 b_2 - a_2 b_1)^2 = \Delta(\mathbf{a}, \mathbf{b})^2$$

Suppose $\mathbf{b} = \lambda \mathbf{a}$ for some scalar λ. Consider $\Delta(\mathbf{a}, \mathbf{b})$:

$$\Delta(\mathbf{a}, \mathbf{b}) = a_1 b_2 - a_2 b_1 = a_1(\lambda a_2) - a_2(\lambda a_1) = \lambda(a_1 a_2 - a_2 a_1) = 0$$

Hence, if $\mathbf{a}$ and $\mathbf{b}$ are both non-zero vectors, then $\Delta(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are scalar multiples of each other. But $\mathbf{a}$ and $\mathbf{b}$ are non-zero and not parallel, hence $\Delta(\mathbf{a}, \mathbf{b}) \neq 0$. Thus $\det \mathbf{T} \neq 0$, hence, $\mathbf{T}$ is invertible, so M_{ij} can be found given $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$, and $\mathbf{q}$; i.e. $\mathbf{M}$ exists.

$$\begin{aligned} &\Delta(\mathbf{a}, \mathbf{b})\mathbf{c} + \Delta(\mathbf{c}, \mathbf{a})\mathbf{b} + \Delta(\mathbf{b}, \mathbf{c})\mathbf{a} \\ &\qquad = (a_1 b_2 - a_2 b_1)\mathbf{c} + (c_1 a_2 - c_2 a_1)\mathbf{b} + (b_1 c_2 - b_2 c_1)\mathbf{a} \\ &\qquad = \begin{pmatrix} a_1 b_2 c_1 - a_2 b_1 c_1 + a_2 b_1 c_1 - a_1 b_1 c_2 + a_1 b_1 c_2 - a_1 b_2 c_1 \\ a_1 b_2 c_2 - a_2 b_1 c_2 + a_2 b_2 c_1 - a_1 b_2 c_2 + a_2 b_1 c_2 - a_2 b_2 c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Hence,

$$\Delta(\mathbf{a}, \mathbf{b})\mathbf{c} + \Delta(\mathbf{c}, \mathbf{a})\mathbf{b} + \Delta(\mathbf{b}, \mathbf{c})\mathbf{a} = \mathbf{0} \tag{1}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Consider $\Delta(\mathbf{p},\mathbf{q}) = \Delta(\mathbf{Ma},\mathbf{Mb})$:

$$\begin{aligned}
\Delta(\mathbf{p},\mathbf{q}) &= \Delta(\mathbf{Ma},\mathbf{Mb}) \\
&= p_1q_1 - p_2q_2 \\
&= (M_{11}a_1 + M_{12}a_2)(M_{21}b_1 + M_{22}b_2) \\
&\qquad - (M_{21}a_1 + M_{22}a_2)(M_{11}b_1 + M_{12}b_2) \\
&= (M_{11}M_{21}a_1b_1 + M_{12}M_{22}a_2b_2 + M_{11}M_{22}a_1b_2 + M_{12}M_{21}a_2b_1) \\
&\qquad - (M_{21}M_{11}a_1b_1 + M_{22}M_{12}a_2b_2 + M_{21}M_{12}a_1b_2 + M_{22}M_{11}a_2b_1) \\
&= (M_{11}M_{22} - M_{12}M_{21})a_1b_2 - (M_{11}M_{22} - M_{12}M_{21})a_2b_1 \\
&= (M_{11}M_{22} - M_{12}M_{21})(a_1b_2 - a_2b_1)
\end{aligned}$$

Hence,

$$\Delta(\mathbf{p},\mathbf{q}) = \Delta(\mathbf{Ma},\mathbf{Mb}) = \det \mathbf{M} \cdot \Delta(\mathbf{a},\mathbf{b}) \tag{2}$$

Suppose that $\mathbf{r} = \mathbf{Mc}$. Then:

$$\begin{aligned}
&\frac{\Delta(\mathbf{a},\mathbf{b})}{\Delta(\mathbf{p},\mathbf{q})} = \frac{\Delta(\mathbf{b},\mathbf{c})}{\Delta(\mathbf{q},\mathbf{r})} = \frac{\Delta(\mathbf{c},\mathbf{a})}{\Delta(\mathbf{r},\mathbf{p})} = \\
&\qquad \frac{\Delta(\mathbf{a},\mathbf{b})}{\Delta(\mathbf{Ma},\mathbf{Mb})} = \frac{\Delta(\mathbf{b},\mathbf{c})}{\Delta(\mathbf{Mb},\mathbf{Mc})} = \frac{\Delta(\mathbf{c},\mathbf{a})}{\Delta(\mathbf{Mc},\mathbf{Ma})} = \det \mathbf{M}
\end{aligned}$$

Therefore,

$$\Delta(\mathbf{a},\mathbf{b}) : \Delta(\mathbf{b},\mathbf{c}) : \Delta(\mathbf{c},\mathbf{a}) = \Delta(\mathbf{p},\mathbf{q}) : \Delta(\mathbf{q},\mathbf{r}) : \Delta(\mathbf{r},\mathbf{p})$$

given that $\mathbf{Mc} = \mathbf{r}$.

Now, suppose that the ratios were equal. Let $\frac{\Delta(\mathbf{a},\mathbf{b})}{\Delta(\mathbf{p},\mathbf{q})} = \frac{\Delta(\mathbf{b},\mathbf{c})}{\Delta(\mathbf{q},\mathbf{r})} = \frac{\Delta(\mathbf{c},\mathbf{a})}{\Delta(\mathbf{r},\mathbf{p})} = \mu$. From (1),

$$\Delta(\mathbf{a},\mathbf{b})\mathbf{c} + \Delta(\mathbf{c},\mathbf{a})\mathbf{b} + \Delta(\mathbf{b},\mathbf{c})\mathbf{a} = \mathbf{0}$$

Multiplying throughout by μ:

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{c} + \Delta(\mathbf{r},\mathbf{p})\mathbf{b} + \Delta(\mathbf{q},\mathbf{r})\mathbf{a} = \mathbf{0}$$

Multiplying throughout by $\mathbf{M}$:

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{Mc} + \Delta(\mathbf{r},\mathbf{p})\mathbf{Mb} + \Delta(\mathbf{q},\mathbf{r})\mathbf{Ma} = \mathbf{0}$$

$\mathbf{Ma} = \mathbf{p}$ and $\mathbf{Mb} = \mathbf{q}$, so:

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{Mc} + \Delta(\mathbf{r},\mathbf{p})\mathbf{q} + \Delta(\mathbf{q},\mathbf{r})\mathbf{p} = \mathbf{0}$$

But from (1) we also know that

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{r}+\Delta(\mathbf{r},\mathbf{p})\mathbf{q}+\Delta(\mathbf{q},\mathbf{r})\mathbf{p}=\mathbf{0}$$

Hence,

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{Mc}+\Delta(\mathbf{r},\mathbf{p})\mathbf{q}+\Delta(\mathbf{q},\mathbf{r})\mathbf{p}=\Delta(\mathbf{p},\mathbf{q})\mathbf{r}+\Delta(\mathbf{r},\mathbf{p})\mathbf{q}+\Delta(\mathbf{q},\mathbf{r})\mathbf{p}=\mathbf{0}$$

Eliminating common terms on both sides:

$$\Delta(\mathbf{p},\mathbf{q})\mathbf{Mc}=\Delta(\mathbf{p},\mathbf{q})\mathbf{r}$$

Since $\Delta(\mathbf{p},\mathbf{q})$ is a scalar, we may eliminate it as a common factor, hence,

$$\mathbf{Mc}=\mathbf{r}$$

Therefore, the condition $\Delta(\mathbf{a},\mathbf{b}):\Delta(\mathbf{b},\mathbf{c}):\Delta(\mathbf{c},\mathbf{a})=\Delta(\mathbf{p},\mathbf{q}):\Delta(\mathbf{q},\mathbf{r}):\Delta(\mathbf{r},\mathbf{p})$ is necessary and sufficient for PQR to be the image of ABC under the transformation $\mathbf{M}$.