$$a_n = \frac{1 + {a_{n-1}}^2}{a_{n-2}}$$

Suppose $a_n = 3a_{n-1} - a_{n-2}$ is true for all $2 \leq n \leq k$ for some $k \geq 2$. Consider a_{k+1}:

$$\begin{aligned}
a_{k+1} &= \frac{1 + {a_k}^2}{a_{k-1}} \\
&= \frac{1 + (3a_{k-1} - a_{k-2})^2}{a_{k-1}} \\
&= \frac{1 + 9{a_{k-1}}^2 - 6a_{k-1}a_{k-2} + {a_{k-2}}^2}{a_{k-1}} \\
&= \frac{1}{a_{k-1}} + 9a_{k-1} - 6a_{k-2} + \frac{{a_{k-2}}^2}{a_{k-1}} \\
&= \frac{1 + {a_{k-2}}^2}{a_{k-1}} + 9a_{k-1} - 6a_{k-2} \\
&= \frac{a_{k-3}}{a_{k-1}} \cdot \frac{1 + {a_{k-2}}^2}{a_{k-3}} + 9a_{k-1} - 6a_{k-2} \\
&= \frac{a_{k-3}}{a_{k-1}} \cdot a_{k-1} + 9a_{k-1} - 6a_{k-2} \\
&= a_{k-3} + 9a_{k-1} - 6a_{k-2} \\
&= -(3a_{k-2} - a_{k-3}) + 9a_{k-1} - 6a_{k-2} + 3a_{k-2} \\
&= -a_{k-1} + 9a_{k-1} - 6a_{k-2} + 3a_{k-2} \\
&= 8a_{k-1} - 3a_{k-2} \\
&= 3(3a_{k-1} - a_{k-2}) - a_{k-1} \\
a_{k+1} &= 3a_k - a_{k-1}
\end{aligned}$$

Hence if $a_n = 3a_{n-1} - a_{n-2}$ is true for all $2 \leq n \leq k$ for some $k \geq 2$, then it is also true for all $n > k$. Consider a_2:

$$a_2 = \frac{1 + {a_1}^2}{a_0} = 2$$

$$a_2 = 3a_1 - a_0 = 2$$

The two formulae agree, so since it is true for $n = 2$, it is also true for all $n \geq 2$.

Suppose $a_n = r^n$ for some non-zero r satisfies the relation $a_n = 3a_{n-1} - a_{n-2}$.

$$r^n = 3r^{n-1} - r^{n-2}$$

$$r^n - 3r^{n-1} + r^{n-2} = 0$$
$$r^2 - 3r + 1 = 0$$
$$r = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = 1 + \frac{1 \pm \sqrt{5}}{2}$$

Let $\beta_+ = 1 + \frac{1+\sqrt{5}}{2}$ and $\beta_- = 1 + \frac{1-\sqrt{5}}{2}$. Any linear combination of ${\beta_+}^n$ and ${\beta_-}^n$ will satisfy the relation $a_n = 3a_{n-1} - a_{n-2}$.

Let us consider some powers of α:

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\begin{aligned}
\alpha^{-1} &= \left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \frac{2}{1+\sqrt{5}} \\
&= \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{2(1-\sqrt{5})}{1-5} = -\frac{2(1-\sqrt{5})}{4} \\
\alpha^{-1} &= -\frac{1-\sqrt{5}}{2} = \alpha - 1
\end{aligned}$$

$$\begin{aligned}
\alpha^2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} \\
&= 1 + \frac{1+\sqrt{5}}{2} = 1 + \alpha
\end{aligned}$$

$$\begin{aligned}
\alpha^{-2} &= \left(-\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} \\
&= 1 + \frac{1-\sqrt{5}}{2} = 1 - \alpha^{-1}
\end{aligned}$$

Therefore, ${\beta_+}^n = \alpha^{2n}$ and ${\beta_-}^n = \alpha^{-2n}$. We also note that $\alpha + \alpha^{-1} = \sqrt{5}$.

Let $a_n = A\alpha^{2n} + B\alpha^{-2n}$ where A and B are coefficients to be determined. $a_1 = 1$, $a_2 = 2$, hence,

$$\begin{cases} A\alpha^2 + B\alpha^{-2} = 1 \\ A\alpha^4 + B\alpha^{-4} = 2 \end{cases}$$

Multiplying the first equation by α^{-2}:

$$A + B\alpha^{-4} = \alpha^{-2}$$

Subtract it from the second equation:

$$\left(A\alpha^4 + B\alpha^{-4}\right) - \left(A + B\alpha^{-4}\right) = 2 - \alpha^{-2}$$

$$A\alpha^4 - A = 2 - \alpha^{-2}$$
$$A\left(\alpha^4 - 1\right) = 2 - \alpha^{-2}$$

$$A = \frac{2-\alpha^{-2}}{\alpha^4 - 1} = \frac{2-(1-\alpha^{-1})}{(\alpha^2+1)(\alpha^2-1)} = \frac{1+\alpha^{-1}}{\alpha(\alpha+\alpha^{-1})(\alpha^2-1)}$$

But $\alpha^{-1} = \alpha - 1$ and $\alpha^2 = 1 + \alpha$, hence,

$$A = \frac{\alpha}{\alpha^2(\alpha+\alpha^{-1})} = \frac{\alpha^{-1}}{\sqrt{5}} \tag{1}$$

$A\alpha^2 + B\alpha^{-2} = \frac{\alpha^{-1}}{\sqrt{5}} \cdot \alpha^2 + B\alpha^{-2} = 1$, hence:

$$\frac{\alpha}{\sqrt{5}} + B\alpha^{-2} = 1$$
$$B\alpha^{-2} = 1 - \frac{\alpha}{\sqrt{5}}$$
$$B = \alpha^2\left(1 - \frac{\alpha}{\sqrt{5}}\right)$$
$$= \alpha^2 \cdot \frac{\sqrt{5}-\alpha}{\sqrt{5}}$$
$$= \alpha^2 \cdot \frac{(\alpha+\alpha^{-1})-\alpha}{\sqrt{5}}$$
$$= \alpha^2 \cdot \frac{\alpha^{-1}}{\sqrt{5}}$$

$$B = \frac{\alpha}{\sqrt{5}} \tag{2}$$

Hence,

$$a_n = \frac{\alpha^{-1}}{\sqrt{5}} \cdot \alpha^{2n} + \frac{\alpha}{\sqrt{5}} \cdot \alpha^{-2n} = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}}$$