UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1201**
ASSESSMENT PATTERN	:	**MATH1201A**
MODULE NAME	:	**Algebra 1**
DATE	:	**10-May-11**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) If $\mathcal{S} = (q \vee \neg p) \bigwedge (q \wedge \neg p)$ find a formula equivalent to $\neg\mathcal{S}$ which does not involve $\wedge, \vee$ or $\neg$

   Without computing truth tables decide which of the following (a), (b) or (c) is correct.

   (a) $\mathcal{S}$ is a tautology , (b) $\mathcal{S}$ is a contradiction ; (c) $\mathcal{S}$ is neither a tautology nor a contradiction.

   (ii) Find a formula equivalent to $\mathcal{T}$ below which does not involve $\exists$, $\neg$, $\vee$ or $\wedge$.

$$\mathcal{T} = \neg(\exists x)\left(\neg P(x) \wedge \left[(\exists y)\neg R(y) \bigvee \neg(\exists z)\neg S(z)\right]\right).$$

   (iii) Explain what is meant by saying that a mapping $f : A \to B$ is *bijective*. Prove that if f is bijective then f is invertible.

   iv) Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 6 & 12 & 14 & 9 & 7 & 11 & 4 & 10 & 5 & 8 & 13 & 1 & 2 & 3 \end{pmatrix}$$

   into a product of disjoint cycles and hence compute $\mathrm{ord}(\sigma)$ and $\mathrm{sign}(\sigma)$.

2. Let $\epsilon(r,s)$ be the $n \times n$ matrix given by $\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. When $n = 1000$ give a formal proof that

$$\epsilon(r,s)\epsilon(777,t) = \begin{cases} \epsilon(r,t) & \text{if } s = 777 \\ 0 & \text{if } s \neq 777. \end{cases}$$

The elementary matrices $\Delta(r,\alpha)$ and $E(r,s;\lambda)$ $(r \neq s)$ are defined by

$$\Delta(r,\alpha) = I_n + (\alpha - 1)\epsilon(r,r) \;;\; E(r,s;\lambda) = I_n + \lambda\epsilon(r,s).$$

When $r \neq s$ express (in terms of I_n, $\epsilon(r,r)$ and $\epsilon(r,s)$) the matrix products

(i) $E(r,s;\mu)\Delta(r,\alpha)$ and (ii) $\Delta(r,\alpha)E(r,s;\lambda)$.

If $\Delta(r,\alpha)E(r,s;\lambda) = E(r,s;\mu)\Delta(r,\alpha)$ express μ in terms of α, λ.

For the matrix A below, find A^{-1}. Moreover, by first expressing A^{-1} as a product of elementary matrices express A also as a product of elementary matrices.

$$A = \begin{pmatrix} -1 & -4 & 1 \\ -2 & -7 & 2 \\ 2 & 3 & -1 \end{pmatrix}.$$

3. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear*.

When T is linear, explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and

(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Find the general solution (over $\mathbb{Q}$) to the system of linear equations $A\mathrm{x} = b$ where

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 3 & 3 & -3 & -3 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}.$$

If $T_A : \mathbb{Q}^7 \to \mathbb{Q}^4$ is the linear mapping $T_A(\mathrm{x}) = A\mathrm{x}$ find also

(i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

4. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V over a field $\mathbb{F}$; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent over $\mathbb{F}$.

In each case below, decide with justification whether the given vectors are linearly independent over $\mathbb{Q}$. If they are not, give an explicit dependence relation between them.

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$;

(b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}$.

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V $(n \geqslant 3)$ and let $\{\mathbf{u}_1, \mathbf{u}_2\} \subset V$ be a linearly independent set. Show that

(i) V has a spanning set of the form $\{\mathbf{u}_1, \mathbf{v}'_2, \mathbf{v}'_3, \ldots, \mathbf{v}'_n\}$ and also that

(ii) V has a spanning set of the form $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}''_3, \ldots, \mathbf{v}''_n\}$

where $\mathbf{v}'_i, \mathbf{v}''_j \in \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for each i, j.

5. Let V be the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) \;=\; a_1 \cos(x) \;+\; a_2 \sin(x) \;+\; a_3\, x \cos(x) + a_4\, x \sin(x) \qquad (a_i \in \mathbb{Q})$$

and let $D : V \to V$ be the linear map $\quad D(f) \;=\; \frac{df}{dx}$.

Taking $\{\, \cos(x)\,,\ \sin(x)\,,\ x\cos(x)\,,\ x\sin(x)\,\}$ as basis for V find :

i) the matrix of D ; ii) the matrix of D^4 ; iii) the matrix of D^{-1}.

Hence *without further explicit differentiation or integration* write down

iv) $\frac{d^4}{dx^4}$ $(\sin(x) - 2\,x\cos(x) + x\sin(x))$ v) $\int \{\cos(x) + x\cos(x) - 2x\sin(x)\}\, dx$
[You may ignore the constant of integration in v)].

6. Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1 \leqslant i \leqslant m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \leqslant j \leqslant n}$ be a basis for V.

   Explain what is meant by the matrix $\mathcal{M}(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right).

   If $S : V \to W$ is also linear and $\Psi = (\psi_k)_{1 \leqslant k \leqslant p}$ is a basis for W prove that

$$\mathcal{M}(S \circ T)^{\Psi}_{\mathcal{E}} = \mathcal{M}(S)^{\Psi}_{\Phi}\mathcal{M}(T)^{\Phi}_{\mathcal{E}}.$$

   Let $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the mapping $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & -x_2 & \\ -x_1 & +x_2 & -x_3 \\ x_1 & +x_2 & +2x_3 \end{pmatrix}$

   and let $\mathcal{E} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ and $\Phi = \left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

   Write down (i) $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $\mathcal{M}(\mathrm{Id})^{\mathcal{E}}_{\Phi}$. Hence find $\mathcal{M}(T)^{\Phi}_{\Phi}$.