UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1402**

ASSESSMENT PATTERN : **MATH1402A**

MODULE NAME : **Mathematical Methods 2**

DATE : **09-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH1402A-001-EXAM-8

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. a) (Chain rule) Let $\omega(t)$ be a composite function

$$\omega(t) = f(X(t),\, Y(t)).$$

Write down the formula for $\omega'(t)$ in terms of $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $X'(t)$, $Y'(t)$.

b) Derive the formula for the chain rule of Part a).
[Hint: You may use that, near $x = x_0, y = y_0$,

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0),$$

and linear Taylor's expansion of $X(t), Y(t)$ near $t = t_0$.]

c) Let R be a region on the xy plane defined by

$$x^2+y^2 \leq 4,\;\; x \geq 0,\, y \geq 0, y \leq x$$

Find the integral

$$\iint_R e^{(x^2+y^2)} y^2 dxdy.$$

2. a) Let $\mathbf{u} = (u_1, u_2, u_3)$ be a unit vector, $|\mathbf{u}| = \sqrt{u_1^2+u_2^2+u_3^2} = 1$. Let $f(x,y,z)$ be a function of 3 variables.
Define the directional derivative $\dfrac{\partial f}{\partial \mathbf{u}}(x_0,y_0,z_0)$.

b) Show that $\dfrac{\partial f}{\partial \mathbf{u}}(x_0,y_0,z_0)$ achieves its maximum, as a function of $\mathbf{u}$, $|\mathbf{u}| = 1$, if

$$\mathbf{u} = \frac{\nabla f(x_0,y_0,z_0)}{|\nabla f(x_0,y_0,z_0)|}$$

c) Let the surface S be given as the graph of the function $f(x,y) = 1 + x^2 + y^2$, where (x,y) satisfy

$$x \geq 0, \quad y \geq 0, \quad x+y \leq 1.$$

Find the surface integral

$$\iint_S \frac{e^{2x+y}}{\sqrt{4z-3}}\, dS.$$

3. a) State the Divergence Theorem carefully.

   b) Verify the Divergence Theorem when the vector-field $\mathbf{F}$ has the form,

$$\mathbf{F}(x,y,z) = a(xy^2\mathbf{i} + yx^2\mathbf{j}) + x^2\cos(\pi z)\mathbf{k}, \quad a > 0,$$

   and D is the cylinder,

$$x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1/2.$$

4. a) State Stoke's Theorem carefully.

   b) Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x,y,z) = xy\mathbf{i} - z\mathbf{j} + y^2\mathbf{k}$$

   and the surface S defined by

$$z - (x^2 + y^2) = 3, \quad z \leq 4.$$

   c) Let R be a triangle on the $xy-$plane with vertices at the points

$$\mathbf{O} = (0,0),\ \mathbf{A} = (a,0),\ \mathbf{B} = (0,b), \quad a,b > 0.$$

   Let $\mathbf{F}$ be a vector field,

$$\mathbf{F}(x,y) = \left(\sin(y)e^{x\sin(y)} + y\right)\mathbf{i} + x\cos(y)e^{x\sin(y)}\mathbf{j}.$$

   Find

$$\oint_C \mathbf{F}\cdot d\mathbf{r},$$

   where C is the boundary of R traversed in the anti-clockwise direction.
   [Hint: You may use Green's Theorem.]

5. a) Let $D = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$. Let C be a smooth curve in D. Let $\mathbf{F}(x, y, z)$ be a smooth vector field in D such that

$$\nabla \times F = 0, \quad \text{in } \mathbf{D}.$$

Does this guarantee that $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the initial, $\mathbf{x}_0 = (x_0, y_0, z_0)$, and terminal, $\mathbf{x}_1 = (x_1, y_1, z_1)$, points of C? Expain your answer.
(Note that $\mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$ is the whole $\mathbb{R}^3$ without the z–axis).

b) Let the vector-field $\mathbf{F}(\mathbf{x})$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, be given by the formula

$$\mathbf{F}(x, y, z) = z\cos(y)e^{xz\cos(y)}\mathbf{i}$$
$$+ \left(1 - xz\sin(y)e^{xz\cos(y)}\right)\mathbf{j} + x\cos(y)e^{xz\cos(y)}\mathbf{k}.$$

Is $\mathbf{F}$ a gradient vector-field and, if so, what is a corresponding potential?

c) Let

$$\mathbf{G} = z\cos(y)e^{xz\cos(y)}\mathbf{i}$$
$$+ \left(1 + xy - xz\sin(y)e^{xz\cos(y)}\right)\mathbf{j} + x\cos(y)e^{xz\cos(y)}\mathbf{k}.$$

Let C be a curve of the form

$$x = \cos(\pi t), \ y = \sin(\pi t), \ z = t(1-t), \quad 0 \le t \le 1,$$

which connects the point $\mathbf{x}_0 = (1, 0, 0)$ with the point $\mathbf{x}_1 = (-1, 0, 0)$.
Find

$$\int_C \mathbf{G} \cdot d\mathbf{r}.$$

[Hint: You may use Part b).]

6. a) State Fourier's Theorem carefully.

b) Find the Fourier coefficients of the function $f(x)$ which is equal to

- π for $-\pi \le x \le 0$;
- $\pi - x$ for $0 \le x < \pi$;
- continued 2π periodically from $(-\pi, \pi)$ to the whole $\mathbb{R}$.

c) Using Part b) or otherwise, show that

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)^2} = \pi^2.$$