UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1302**

ASSESSMENT PATTERN : **MATH1302A**

MODULE NAME : **Applied Mathematics 2**

DATE : **26-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

1. A particle of mass m is projected with velocity $\underline{U}$ from the origin at time $t = 0$. It moves under the action of the gravity force $m\underline{g}$ and of air resistance, which is proportional to its speed.

   Write down the vector equation of motion for this system, defining all the quantities you use.

   Find the particle's velocity and position as a function of time. If the particle is initially projected horizontally, show that its horizontal range (if it is allowed to fall indefinitely) is

$$x_{\max} = \frac{|\underline{U}|\,|\underline{v}_T|}{|\underline{g}|}$$

   where $\underline{v}_T$ is the particle's terminal velocity.

2. A point particle of mass m moves under gravity on a curve γ, which lies in a vertical plane, with s denoting distance along γ and $\psi(s)$ the (variable) angle of inclination from the horizontal at each position along γ.

   (a) Write down expressions for the unit tangent vector $\underline{e}_t$ and the unit normal vector $\underline{e}_n$ in terms of the standard basis vectors $\underline{i}$ and $\underline{j}$.

   (b) Give the tangential and normal components of velocity, and show that those of of acceleration are $\ddot{s}$, $\kappa\dot{s}^2$ respectively, where $\kappa = \mathrm{d}\psi/\mathrm{d}s$.

   (c) If the reaction force R is normal to the curve, justify the governing equations $\ddot{s} = -g\sin\psi$, $m\kappa\dot{s}^2 = R - mg\cos\psi$ for the motion.

   (d) If, also, γ is defined by $s^3 = \sin\psi$, with $0 < \psi < \pi/2$, and initially $s = s_0$, $\dot{s} = 0$, solve the tangential governing equation to give $\dot{s}^2$ as a function of s only. Deduce the speed of the particle when it reaches the point $s = 0$.

3. Write down the formulae for acceleration in the radial, tangential and vertical directions in cylindrical polar coordinates $\{\rho, \theta, z\}$.

   A particle of mass m moves on the inside of a smooth parabolic bowl specified in cylindrical coordinates as $z = \rho^2$. If ψ denotes the angle the bowl makes with the horizontal, write down the equations governing the motion and show that angular momentum
$$h = \rho^2\dot{\theta}$$
   is conserved.

   Derive the energy equation for this system. If the particle is initially projected horizontally with speed $V \geq \sqrt{2g}$ from a point with $\rho = z = 1$, show that
$$\tfrac{1}{2}(\dot{\rho}^2 + V^2/\rho^2 + 4\rho^2\dot{\rho}^2) + g\rho^2 = \tfrac{1}{2}V^2 + g,$$
   and deduce that the particle is restricted to the region
$$1 \leq z \leq V^2/2g.$$
   What is the particle's motion if $V = \sqrt{2g}$?

4. Write down the radial and tangential components of acceleration in terms of polar coordinates r, θ for a particle moving in a plane.

   A particle of mass m moves under a force $f(r)\underline{\hat{r}}$, where $\underline{\hat{r}}$ is a unit vector in the direction of the radius vector $\underline{r}$ from the origin $\underline{0}$ to the particle, and $r = |\underline{r}|$. Show that $u = 1/r$ satisfies
$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{f(1/u)}{mh^2u^2},$$
   where h is a constant.

   A particle of mass m moves under the action of an attractive central force of constant magnitude k. Give conditions on b and h such that a circular orbit of radius b is possible.

   Show that such an orbit is stable to small perturbations which preserve h.

5. A rocket of mass M is initially carrying a mass m_0 of fuel. It lifts off from a stationary position on the ground at time $t = 0$. It burns fuel at a rate α per unit time, and the spent fuel is propelled backwards relative to the rocket with relative velocity V. The rocket moves under gravity only: neglect air resistance.

   Write down the equations governing the rocket's motion. Find the mass of fuel as a function of time, and the rocket's velocity as a function of time. What is the velocity at the moment when all fuel is spent?

6. The tension T in an elastic spring of negligible mass is given by $T = k(l_l - l)$, where l is its natural length, l_1 is its stretched length and $k > 0$ is a stiffness constant.

   Three identical elastic springs AB, BC and CD with stiffness constant k and natural length l are attached together vertically between two horizontal plates a distance $3l$ apart, so that end A is attached to the upper plate and end D to the lower plate. A particle of mass m is attached to the point B joining the upper two springs, and another (also of mass m) to the point C joining the lower two springs.

   (a) Determine the equilibrium lengths of the three springs when the system hangs under gravity without moving.

   (b) If x is the displacement of point B from equilibrium and y the displacement of point C from equilibrium, show that

   $$\begin{aligned} m\ddot{x} &= k(y - 2x) \\ m\ddot{y} &= k(x - 2y) \end{aligned}$$

   (c) Find expressions for the general solution $x(t)$, $y(t)$ of the system.