1

If ak^{-1}, a, and ak are roots, then,

$$x^3 - px^2 + qx - r = \left(x - ak^{-1}\right)(x - a)(x - ak) \tag{1}$$

$$x^3 - px^2 + qx - r = x^3 - a\left(k^{-1} + 1 + k\right)x + a^2\left(k^{-1} + 1 + k\right)x^2 - a^3$$

Comparing coefficients,

$$\begin{aligned} p &= a\left(k^{-1} + 1 + k\right) && (2) \\ q &= a^2\left(k^{-1} + 1 + k\right) && (3) \\ r &= a^3 && (4) \end{aligned}$$

$$q^3 - rp^3 = a^6\left(k^{-1} + 1 + k\right)^3 - a^3 \cdot a^3\left(k^{-1} + 1 + k\right)^3 = 0$$

Hence, $q/p = a$ is a root and $q^3 - px^3 = 0$, if ak^{-1}, a, and ak are roots.

2

Let $r = q^3/p^3$. Then,

$$x^3 - px^2 + qx - r = x^3 - px^2 + qx - \frac{q^3}{p^3} = 0 \tag{5}$$

Let $x = q/p$.

$$\left(\frac{q}{p}\right)^3 - p\left(\frac{q}{p}\right)^2 + q\left(\frac{q}{p}\right) - \frac{q^3}{p^3} = \frac{q^3}{p^3} - \frac{q^2}{p} + \frac{q^2}{p} - \frac{q^3}{p^3} = 0$$

Hence q/p is a root if $r = q^3/p^3$. But r is the product of the roots, hence the product of the other two roots must be $(q/p)^2$. Let us denote the roots by α, β, γ and let $\beta = q/p$. Thus, $\alpha\gamma = q^2/p^2$. Let k be a constant such that $\gamma = k^2\alpha$. Then $\alpha\gamma = k^2\alpha^2 = q^2/p^2$. Since we can choose the sign of k, let k be such that $\beta = q/p = k\alpha$. Hence the roots are $\alpha, k\alpha, k^2\alpha$, which forms a geometric progression.

3

Let α, β, γ be the roots of the equation. Then,

$$x^3 - px^2 + qx - r = (x - \alpha)(x - \beta)(x - \gamma) = 0 \tag{6}$$

Expanding and comparing coefficients, we find:

$$\begin{aligned} p &= \alpha + \beta + \gamma && (7) \\ q &= \alpha\beta + \beta\gamma + \gamma\alpha && (8) \\ r &= \alpha\beta\gamma && (9) \end{aligned}$$

Suppose $\beta - \alpha = \gamma - \beta = \Delta$. Then $\alpha = \beta - \Delta$ and $\gamma = \beta + \Delta$. Thus:

$$p = \alpha + \beta + \gamma = 3\beta \tag{10}$$

$$\begin{aligned} q &= \alpha\beta + \beta\gamma + \gamma\alpha = \beta(\beta - \Delta) + \beta(\beta + \Delta) + (\beta - \Delta)(\beta + \Delta) \\ &= \left(\beta^2 - \Delta\beta\right) + \left(\beta^2 + \Delta\beta\right) + \left(\beta^2 - \Delta^2\right) = 3\beta^2 - \Delta^2 \end{aligned} \tag{11}$$

$$r = \alpha\beta\gamma = \beta(\beta - \Delta)(\beta + \Delta) = \beta\left(\beta^2 - \Delta^2\right) \tag{12}$$

$$3q = 9\beta^2 - 3\Delta^2 = p^2 - 3\Delta^2$$

$$27\frac{r}{p} = \frac{1}{3\beta} \cdot 27\beta\left(\beta^2 - \Delta^2\right) = 9\beta^2 - 9\Delta^2 = p^2 - 9\Delta^2$$

$$9\Delta^2 = 3\left(p^2 - 3q\right) = p^2 - 27\frac{r}{p}$$

$$2p^2 - 9q + 27\frac{r}{p} = 0$$

$$2p^3 - 9pq + 27r = 0 \tag{13}$$

Hence (13) is necessary for the roots to be in arithmetic progression. Because $p \neq 0$, the argument is also valid in reverse, so (13) is also sufficient.