$\displaystyle\frac{1}{n}\sum_{m=1}^{n} f\left(1+\frac{m}{n}\right)$ gives an approximation of the (signed) area under the curve $y = f(x)$ between $x = 1$ and $x = 1 + \frac{n}{n} = 2$ by approximating the area of strips of width $1/n$ by rectangles. As $n \to \infty$, the strips become infinitesimally small, so the limit of the sequence is the area under the curve, which is given by $\displaystyle\int_1^2 f(x)\,dx$. (Diagram omitted.)

# 1

$$\begin{aligned}
\lim_{n\to\infty} &\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{n+n}\right)\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{n}{n+m}\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{1+\frac{m}{n}}\right\} \\
&= \int_1^2 \frac{1}{x}\,dx = \Big[\ln x\Big]_1^2 = \ln 2
\end{aligned}$$

# 2

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+n^2}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{n^2}{n^2+1}+\frac{n^2}{n^2+4}+\cdots+\frac{n^2}{n^2+n^2}\right)\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{n^2}{n^2+m^2}\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{1+\left(\frac{m}{n}\right)^2}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\left(1+\frac{m}{n}\right)^2-2\left(\frac{m}{n}\right)}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\left(1+\frac{m}{n}\right)^2-2\left(1+\frac{m}{n}\right)+2}\right\} \\
&= \int_1^2 \frac{1}{x^2-2x+2}\,dx = \int_0^1 \frac{1}{u^2+1}\,du = \int_{\arctan 0}^{\arctan 1}\frac{(\sec\theta)^2}{(\tan\theta)^2+1}\,d\theta \\
&= \int_0^{\frac{1}{4}\pi}\frac{(\sec\theta)^2}{(\sec\theta)^2}\,d\theta = \int_0^{\frac{1}{4}\pi} 1\,d\theta = \frac{1}{4}\pi
\end{aligned}$$