Question 24 (*****)

The point P lies on the ellipse has polar equation

$$r(5-3\cos\theta)=8,\ 0\le\theta<2\pi.$$

The ellipse has foci at $O(0,0)$ and at $T(3,0)$.

Show that $|OP|+|PT|$ is constant for all positions of P.

$$\boxed{|OP|+|PT|=5}$$

$r(5-3\cos\theta)=8$

$\Rightarrow r=\frac{8}{5-3\cos\theta}$

$\Rightarrow r=\frac{8}{5-3\left(\frac{x}{r}\right)}$

MULTIPLY TOP & BOTTOM BY r

$\Rightarrow r=\frac{8r}{5r-3x}$

$\Rightarrow 1=\frac{8}{5r-3x}$

$\Rightarrow 5r-3x=8$

$\Rightarrow 5r=8+3x$

$\Rightarrow 25r^2=64+48x+9x^2$

$\Rightarrow 25x^2+25y^2=64+48x+9x^2$

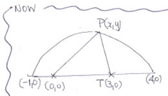
$\Rightarrow 16x^2-48x+25y^2=64$

$\Rightarrow x^2-3x+\frac{25}{16}y^2=4$

$\Rightarrow \left(x-\frac{3}{2}\right)^2-\frac{9}{4}+\frac{25}{16}y^2=4$

$\Rightarrow 16\left(x-\frac{3}{2}\right)^2+25y^2=100$

NOW



$d=|OP|+|TP|$

$d=\sqrt{x^2+y^2}+\sqrt{(x-3)^2+y^2}$

$d=\sqrt{x^2+y^2}+\sqrt{x^2+y^2-6x+9}$

NOW $25y^2=100-16\left(x-\frac{3}{2}\right)^2$

$y^2=4-\frac{16}{25}\left(x^2-3x+\frac{9}{4}\right)$

$y^2=4-\frac{16}{25}x^2+\frac{48x}{25}-\frac{36}{25}$

$y^2=\frac{64}{25}+\frac{48}{25}x-\frac{16}{25}x^2$

HENCE $d=\sqrt{x^2+\left(\frac{64}{25}+\frac{48}{25}x-\frac{16}{25}x^2\right)}+\sqrt{x^2+\left(\frac{64}{25}+\frac{48}{25}x-\frac{16}{25}x^2\right)-6x+9}$

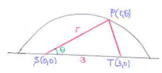
$d=\sqrt{\frac{9}{25}x^2+\frac{48}{25}x+\frac{64}{25}}+\sqrt{\frac{9}{25}x^2-\frac{102}{25}x+\frac{289}{25}}$

$d=\frac{1}{5}\sqrt{9x^2+48x+64}+\frac{1}{5}\sqrt{9x^2-102x+289}$

$d=\frac{1}{5}\sqrt{(3x+8)^2}+\frac{1}{5}\sqrt{(17-3x)^2}\quad(-1\le x\le 4)$

$d=\frac{1}{5}(3x+8)+\frac{1}{5}(17-3x)=5$

ALTERNATIVE IN POLARS



$r=\frac{8}{5-3\cos\theta}$

$|OP|+|PT|=r+|PT|$

$=r+\sqrt{|SP|^2+|ST|^2-2|SP||ST|\cos\theta}$ (COSINE RULE)

$=r+\sqrt{r^2+9-2\times3\times r\cos\theta}$

$=r+\sqrt{r^2-6r\cos\theta+9}$

REARRANGE THE EQUATION OF THE CURVE

$5-3\cos\theta=\frac{8}{r}$

$5-\frac{8}{r}=3\cos\theta$

$5r-8=3r\cos\theta$

$16-10r=-6r\cos\theta$

ALSO NOTE HERE

$-1\le\cos\theta\le1$

$2\le5-3\cos\theta\le8$

$1\le\frac{8}{5-3\cos\theta}\le4$

$1\le r\le4$

$\ldots=r+\sqrt{r^2+(16-10r)+9}$

$=r+\sqrt{r^2-10r+25}$

$=r+\sqrt{\cancel{(r-5)^2}}$

OR

$r+\sqrt{(5-r)^2}$

$=r+(5-r)$

$=5$