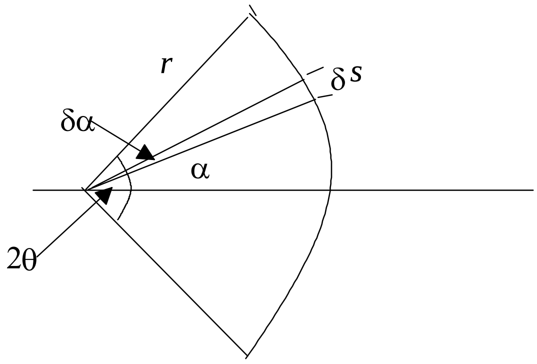
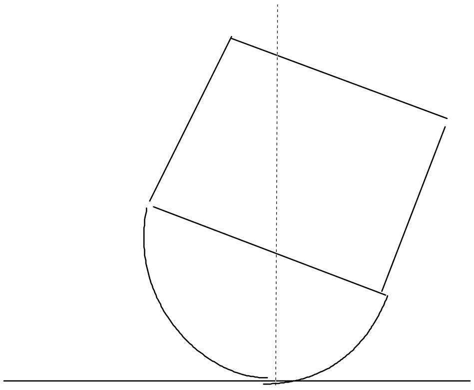
11. Consider an element of the arc of length δs as shown
Take axis of symmetry of arc as the x axis
and let $(\bar{x}, 0)$ be the position of the centre of mass.
Let mass of complete arc be M then taking moments about y axis

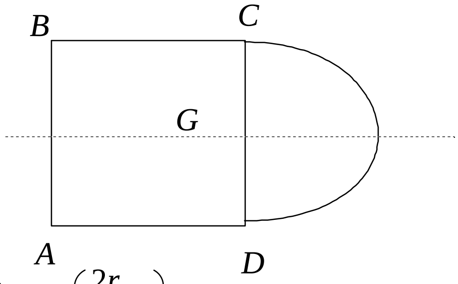


$$M\bar{x} = \sum_{\alpha=-\theta}^{\theta} \frac{r\delta\alpha}{2r\theta} M.r\cos\alpha = \frac{Mr}{2\theta}\int_{-\theta}^{\theta} \cos\alpha \, d\alpha$$

$$\Rightarrow \bar{x} = \frac{r}{2\theta}[\sin\alpha]_{-\theta}^{\theta} = \frac{r\sin\theta}{\theta}$$



If the framework can be in equilibrium with any part of the semicircle in contact with the floor then the centre of mass of the framework must be directly above this point of contact hence at centre of semicircular arc.



Taking axis of symmetry as x axis and centre of arc as origin
we have, assuming a mass m per unit of length
a mass $4m(r+h)$ with cof m at $(-h, 0)$ and a mass πrm with c of m at $\left(\frac{2r}{\pi}, 0\right)$
so for centre of mass of whole to be at the origin we must have $4mh(r+h) = \pi rm.\frac{2r}{\pi}$

$$\Rightarrow 4hr + 4h^2 = 2r^2 \Rightarrow 2h^2 + 2hr - r^2 = 0 \Rightarrow h = \frac{1}{4}\left(-2r + \sqrt{4r^2 + 8r^2}\right) = \frac{1}{2}(\sqrt{3} - 1)r$$

we can obviosly ignore the negative root.

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12. If applying force directly backwards then equation of motion is

$$Mv\frac{dv}{dx} = -\frac{F(V^2+v^2)}{2V^2} \Rightarrow \int \frac{Mv\,dv}{V^2+v^2} = \int \frac{-F}{2V^2}dx \Rightarrow \frac{M}{2}\ln(V^2+v^2) = -\frac{Fx}{2V^2} + c$$

$x = 0$ when $v = V$ so $c = \frac{M}{2}\ln 2V^2$, so $\ln\left(\frac{V^2+v^2}{2V^2}\right) = -\frac{Fx}{MV^2}$ and $v = 0$ when $x = \frac{MV^2}{F}\ln 2$

So to avoid hitting the bank we must have $x < d \Rightarrow d > \frac{MV^2}{F}\ln 2$

If applying force at right angles to velocity, $F = \frac{Mv^2}{r}$ where r is the radius of the circle she will describe so again to avoid hitting the bank we must have $r < d \Rightarrow d > \frac{MV^2}{F}$

Hence, she can avoid hitting the bank if $d > \frac{MV^2}{F}$

The assumption that she is a particle is probably o.k. for the first case but in the second case she is rotating and this would rerally need to be taken into account.

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13. When hanging in equilibrium, the tension in each rope T is given by $T = \frac{490000d}{10}$
where d is the extension and resolving vertically $2T = 1000g$ so $49000d = 500g \Rightarrow d = \frac{500g}{49000} = 0.1$ m
By conservation of momentum, if speed of plank and Nelly is V after she lands then
$5000V = 4000 \times 5 \Rightarrow V = 4\ \text{ms}^{-1}$
If initial height of plank above floor was h m then by work-energy principle
loss of kinetice energy $= \frac{1}{2} \times 5000 \times 4^2 = 40000$, loss of gravitational p.e. $= 5000gh$

elastice energy before Nelly lands on plank $= 2\times\frac{1}{2}.4900\times 0.1 = 490$

elastic energy when plank comes to rest $= 2\times\frac{1}{2}\times\frac{490000h^2}{10}$ wher h is new stretched length of ropes

Gain of elastic energy $= 49000h^2 - 490$

hence, $49000h^2 - 490 = 40000 + 49000(h-0.1) \Rightarrow 49000h^2 - 49000h - 35590 = 0$

$\Rightarrow 4.9h^2 - 4.9h - 3.559 = 0 \Rightarrow h = \frac{1}{9.8}\left(4.9 + \sqrt{4.9^2 + 4\times 4.9\times 3.559}\right) = \frac{4.9+9.68}{9.8} = 1.488$

so initial height above the floor is 1.388 m

Nelly now steps off the plank so if x is the distance of the plank below its equilibrium position then ropes will go slack if $49000h^2 > 1000gh$ i.e .if $49000h > 1000g$

$49000h = 72900$ and $1000g = 9800$ so the ropes will become slack

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14. $\mathrm{E}\left[X_M\right] = \int_{-\infty}^{M} x\phi(x)\,\mathrm{d}x + \int_{M}^{\infty} M\phi(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} x\exp\left(-\frac{1}{2}(x-1)^2\right)\mathrm{d}x + M(1-\Phi(M))$

$= \left[-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x-1)^2\right)\right]_{-\infty}^{M} + M(1-\Phi(M)) = -\phi(M) + M(1-\Phi(M))$ as required

If X is the number of potential bus passengers then $X \sim \mathrm{B}(1024, 0.5) \approx \mathrm{N}\left(512, 16^2\right)$

Profit in vloskas per occasion is simply the number of passengers carried

Extra profit from an extra bus is 0 if $X < 496$, $X-496$ if $496 < X < 527$ and 31 if $X \ge 527$

let $X' = X - 496$ then extra profit is X' if $X' < 31$ and 31 if $X' \ge 31$

Using above result, expected extra profit $= -\phi\left(\frac{31}{16}\right) + 31\left(1-\Phi(\frac{31}{16})\right) = -\phi(-1) + 31(1-\Phi(31))$

$= -0.26 + 31\times(1-0.834) = 4.9$ which occurs 50 times a year so maximum bribe would be £244

Expected extra profit $= \sum_{k=496}^{527}(k-496)\phi\left(-1+\frac{k}{16}\right) + 31\times\left(1-\Phi\left(1-\frac{15}{16}\right)\right)$

Expected lost profit ($X > 496$) without another bus is $\sum_{k=1}^{31} k\mathrm{P}(X = 496+k) + 31\mathrm{P}(X > 527)$

i.e. $\sum_{k=1}^{31}\frac{k}{16}\phi\left(-1+\frac{k}{16}\right) \approx \int_{1}^{31}\frac{k}{16}\phi\left(-1+\frac{k}{16}\right)\mathrm{d}x = \int_{1}^{31}\frac{x}{16}\cdot\frac{1}{\sqrt{2\pi}}\phi\left(-1+\frac{x}{16}\right)\mathrm{d}x$

$= \int_{1}^{31}\frac{x}{16}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)\mathrm{d}x$ (where $z = -1+\frac{x}{16}$)

$= \int_{1}^{31}\frac{x}{16}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-31)^2}{512}\right)\mathrm{d}x$

now let $t = \frac{x-31}{16} \Rightarrow \mathrm{d}t = \frac{1}{16}\mathrm{d}x,\ x = 0 \Rightarrow t = -\frac{31}{16},\ x = 31 \Rightarrow t = 0$ so integral becomes

$\int_{-31/16}^{0}\frac{t+\frac{31}{16}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t = \int_{-2}^{0}\frac{t}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t + \frac{31}{16}\int_{-2}^{0}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t$

$= 16\left[-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\right]_{-2}^{0} + 32\left[\frac{1}{2} - \Phi(-2)\right] + 16 = \frac{16}{\sqrt{2\pi}}\left(\mathrm{e}^{-2}-1\right) + 32\Phi(2) - 16 + 16$

in the course of a year the expected loss is 50 time this i.e. $1600\Phi(2) - \frac{800}{\sqrt{2\pi}}\left(1-\mathrm{e}^{-2}\right)$

which is thus the largest annual bribe they should consider paying

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