STEP Mathematics Paper I 1988

16.p.d.f. of t is
$$\frac{kt^2}{(1+t^2)^2}$$
 for some k so we must have $\int_0^{\infty} \frac{kt^2}{(1+t^2)^2} dt = 1$
let $t = \tan \theta$ so integral becomes $k \int_0^{\pi/2} \left(\frac{\tan^2 \theta}{\sec^2 \theta} \cdot \sec^2 \theta d\theta\right) = k \int_0^{\pi/2} \sin^2 \theta d\theta = k \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta) d\theta$
 $= \frac{k}{2} \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{k\pi}{4} = 1$ so $k = \frac{4}{\pi} \Rightarrow p.d.f$ is $\frac{4t^2}{\pi(1+t^2)^2}$
So P(car becomes rusty before it is $3 + t$ years old) $= \int_0^t \frac{4x^2}{\pi(1+x^2)^2} dx = \frac{2}{\pi} \left[\tan^{-1}t - \frac{t}{1+t^2}\right]$
P(rusty before 4 years old) $= \frac{2}{\pi} \left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.1817 = p$ say
P(rusty before 5 years old) $= \frac{2}{\pi} \left(\tan^{-1}2 - \frac{2}{5}\right) = 0.4502 = q$ say
P(car fails on 5th anniverary) = P(passes on 4th and fails on 5th)
P(not rusty at 4 years but rusty at 5 years and then fails) $= \frac{q-p}{2}$
P(rusty at 4 years but passes then fails at 5 years) $= \frac{p}{2} \times \frac{1}{2} = \frac{p}{4}$
so P(fails on 5th anniversary) $= \frac{q}{2} - \frac{p}{4} = 0.2251 - 0.0454 = 0.2056$
P(rusty at 4 years | destroyed at 5 years) $= \frac{\frac{p}{4}}{\frac{q}{2} - \frac{p}{4}} = \frac{p}{2q-p} = \frac{0.1817}{0.9004 - 0.1817} = 0.2516$