Examination Style Paper

1 An ellipse E has a focus at (6, 0) and the corresponding directrix has equation $x = 12$, find

a the exact value of the eccentricity of E, **(4)**

b a Cartesian equation of E. **(2)**

2
$$\mathrm{f}(x) = \frac{\cosh 2x}{\sqrt{1+\sinh 2x}}, \qquad x > 0$$

Find the x- coordinate of the stationary point of f(x), giving your answer to 3 significant figures. **(7)**

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3 **a** Find $\int \frac{1+6x}{\sqrt{1-9x^2}}\,\mathrm{d}x$ **(5)**

b Find the exact value of $\int_0^{\frac{1}{6}} \frac{1+6x}{\sqrt{1-9x^2}}\,\mathrm{d}x$ **(3)**

4
$$I_n = \int x^n \mathrm{e}^{3x}\,\mathrm{d}x, \quad n \geq 0$$

a Prove that, for $n \geq 1$,

$$I_n = \frac{1}{3}\left(x^n \mathrm{e}^{3x} - nI_{n-1}\right)$$ **(3)**

b Find, in terms of e, the exact value of

$$\int_0^2 x^3 \mathrm{e}^{3x}\mathrm{d}x$$ **(5)**

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5 The line l_1 has Cartesian equations $x - 2 = 4y = z + 3$ and the line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

The plane Π_1 contains l_1 and l_2.

a Find a vector which is normal to Π_1. **(3)**

b Show that an equation of Π_1 is $8x + 4y - 9z = 43$ **(2)**

c Find the shortest distance of the point D (1, 1, 1) from Π_1. **(3)**

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6 **Figure 1**

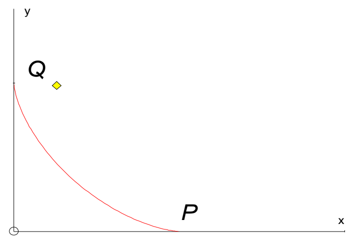

Figure 1 shows a sketch of the curve with parametric equations

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \le t \le \frac{\pi}{2}$$

where a is a positive constant.
The curve cuts the x-axis at the point P and the y axis at the point Q.
Find the perimeter of the shaded region bounded by the arc PQ and the chord PQ. **(8)**

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7 The point $P(ap^2, 2ap)$ lies on the parabola M with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to M at P is

$$py = x + ap^2$$ **(3)**

The point $Q\ (9ap^2, 6ap)$ also lies on M.

b Write down an equation of the tangent to M at Q. **(2)**

The tangent at P and the tangent at Q intersect at the point V.

c Find, as p varies, the locus of V. **(4)**

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8 **a** Using the definition of $\sinh x$ or $\operatorname{cosech} x$ in terms of exponentials, show that, for $x \ge 0$

$$\operatorname{arcosech} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$$ **(5)**

b Solve the equation

$$3\coth^2 x = 7\operatorname{cosech} x + 1$$

giving your answers in terms of natural logarithms. **(5)**

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9 $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ k & 2 & 5 \\ l & 2 & -1 \end{pmatrix}$ where k and l are constants

a Show that $\begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$ is an eigenvector of **A** and find the corresponding eigenvalue. **(2)**

Given that -3 is also an eigenvalue of **A**

b find the corresponding eigenvector. **(4)**

Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is also an eigenvector of **A**

c find the corresponding eigenvalue, the value of k and the value of l. **(5)**

Worked Solutions

1 **a** $ae = 6$ and $\dfrac{a}{e} = 12$

Use formula from booklet for focus and directrix — B1 B1

Solving $a^2 = ae \times \dfrac{a}{e} = 72$

So $e = \dfrac{6}{\sqrt{72}} = \dfrac{1}{\sqrt{2}}$ — M1 A1

b $b^2 = 72\left(1 - \tfrac{1}{2}\right) = 36$

Use $b^2 = a^2\left(1 - e^2\right)$ — M1

So equation of E is $\dfrac{x^2}{72} + \dfrac{y^2}{36} = 1$

Use the standard equation of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ — A1

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2 $$\mathrm{f}'(x) = \frac{\sqrt{1+\sinh 2x} \times 2\sinh 2x - \cosh 2x \times \tfrac{1}{2}\left(1+\sinh 2x\right)^{-\frac{1}{2}} \times 2\cosh 2x}{\left(1+\sinh 2x\right)}$$ — M1A1A1

$$\mathrm{f}'(x) = \frac{2\sinh 2x\left(1+\sinh 2x\right) - \cosh^2 2x}{\left(1+\sinh 2x\right)^{\frac{3}{2}}}$$

Differentiate using the quotient rule and then simplify. Then set derivative = 0

$\mathrm{f}'(x) = 0 \Rightarrow 0 = \sinh^2 2x + 2\sinh 2x - 1$

Use $\cosh^2 A - \sinh^2 A = 1$ — M1 A1

So $\sinh 2x = -1 \pm \sqrt{2}$

and $x = \dfrac{1}{2}\operatorname{arsinh}(\sqrt{2} - 1)$

$= 0.201599\ldots = 0.202$ (3sf)

Solve using the quadratic formula or complete the square — M1 A1

3	**a**	$I = \int \frac{1}{\sqrt{1-9x^2}}\,dx + \int 6x\left(1-9x^2\right)^{-\frac{1}{2}}\,dx$	Split the integral into two parts	M1
	So	$I = \frac{1}{3}\arcsin 3x - \frac{2}{3}\left(1-9x^2\right)^{\frac{1}{2}} + c$	First integral is based on arcsin$x$ given in formula booklet. For second identify $f'(x)f(x)$ or use substitution	M1A1 M1A1
	b	$I = \left[\frac{1}{3}\arcsin 3x - \frac{2}{3}\left(1-9x^2\right)^{\frac{1}{2}}\right]_0^{\frac{1}{6}}$		
		$= \left(\frac{1}{3}\arcsin\left(\frac{3}{6}\right) - \frac{2}{3}\left(1-\frac{9}{36}\right)^{\frac{1}{2}}\right) - \left(0 - \frac{2}{3}\right)$	Substitute limits and recall that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	M1
		$= \frac{\pi}{18} - \frac{\cancel{2}}{3} \times \frac{\sqrt{3}}{\cancel{2}} + \frac{2}{3}$		
		$= \frac{\pi}{18} - \frac{\sqrt{3}}{3} + \frac{2}{3}$		A1 A1
4	**a**	$I_n = \int x^n \, d\left(\frac{1}{3}e^{3x}\right) = \frac{1}{3}e^{3x}x^n - \int \frac{1}{3}e^{3x}nx^{n-1}\,dx$	Use integration by parts	M1A1
		$I_n = \frac{1}{3}\left(x^n e^{3x} - nI_{n-1}\right)$	Simplify to given answer	A1cso
	b			
		$I_3 = \frac{1}{3}x^3e^{3x} - I_2$		
		$I_2 = \frac{1}{3}x^2e^{3x} - \frac{2}{3}I_1$	Use the reduction formula from (a)	M1
		$I_1 = \frac{1}{3}xe^{3x} - \frac{1}{3}I_0$		
		$I_0 = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$	Use integration to find the I_0 term	B1
	So	$I_3 = \frac{1}{3}x^3e^{3x} - \frac{1}{3}x^2e^{3x} + \frac{2}{9}xe^{3x} - \frac{2}{9}\times\frac{1}{3}e^{3x}$	Link the terms together	M1
		$\int_0^2 x^3e^{3x}\,dx = \left[\frac{1}{3}x^3e^{3x} - \frac{1}{3}x^2e^{3x} + \frac{2}{9}xe^{3x} - \frac{2}{9}\times\frac{1}{3}e^{3x}\right]_0^2$	Apply the limits	M1
		$= \left(\frac{8}{3}e^6 - \frac{4}{3}e^6 + \frac{4}{9}e^6 - \frac{2}{27}e^6\right) - \left(-\frac{2}{27}\right)$		
		$= \frac{46}{27}e^6 + \frac{2}{27}$		A1

5 **a** l_1 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$ — B1for 1:0.25:1 ratio

Perpendicular vector to plane is $\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 9 \end{pmatrix}$ — Use the vector product — M1A1

b Since plane contains both lines:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} -8 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 9 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = -43$$

Use **r.n=a.n** formula for a plane — M1 A1cso

So $-8x - 4y + 9z = -43$
Or $8x + 4y - 9z = 43$

c Vector from $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$ — Find a suitable vector and then scalar product with $\hat{\mathbf{n}}$ — M1

Distance is $\pm \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} -8 \\ -4 \\ 9 \end{pmatrix} \times \dfrac{1}{\sqrt{(-8)^2 + (-4)^2 + 9^2}}$ — M1

So shortest distance is $\dfrac{40}{\sqrt{161}}$ — A1

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6 $P\ (a, 0)$ and $Q\ (0, a)$ so chord $PQ = a\sqrt{2}$ — B1

Arc $PQ = \int \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t$ — Use formula from formula booklet

$\dot{x} = 3a\cos^2 t \times (-\sin t)$

$\dot{y} = 3a\sin^2 t \times (\cos t)$ — M1 A1

So arc $= \int 3a\sin t\cos t\sqrt{\sin^2 t + \cos^2 t}\ \mathrm{d}t$ — M1

$= \int \dfrac{3a}{2}\sin 2t\ \mathrm{d}t$ — Use sin2A formula to help integrate — M1

$= \left[-\dfrac{3a}{4}\cos 2t\right]_0^{\frac{\pi}{2}}$ — A1

$= \left(\dfrac{3a}{4}\right) - \left(-\dfrac{3a}{4}\right)$

So perimeter $= \dfrac{3a}{2} + a\sqrt{2}$ — M1 A1

7	**a** $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2a}{2ap} = \frac{1}{p}$	Differentiate using chain rule to find gradient	M1
	Equation is $y - 2ap = \frac{1}{p}\left(x - ap^2\right)$ So $py = x + ap^2$		M1 A1cso
	b Let $q = 3p$ So equation is: $3py = x + 9ap^2$	Let Q be ($aq^2, 2aq$) and then compare with given point	M1 A1
	c Subtracting equations $2py = 8ap^2$ So $y = 4ap$ And $x = 3ap^2$	Solve simultaneously to find the coordinates of V	M1 A1
	Therefore $y^2 = 16a^2p^2$ and $3y^2 = 48a^2p^2$ So locus of V is: $3y^2 = 16ax$	Eliminate p to find locus	M1 A1
8	**a** Let $y = \text{arcosech}x$ Then $x = \text{cosech}y$		M1
	So $\sinh y = \frac{1}{x}$ $\frac{e^y - e^{-y}}{2} = \frac{1}{x}$	Use the definition of sinh$x$	M1
	$xe^{2y} - x = 2e^y$ or $xe^{2y} - 2e^y - x = 0$	Rearrange to form quadratic in e^y	A1
	$e^y = \frac{2 \pm \sqrt{4 + 4x^2}}{2x} = \frac{1 \pm \sqrt{1 + x^2}}{x}$	Use the quadratic formula	M1
	Since $e^y > 0$ then we must choose the "+" sign So $y = \text{arcosech}x = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$	Take logs	A1cso
	b $3 + 3\text{cosech}^2 x = 7\text{cosech}x + 1$ $3c^2 - 7c + 2 = 0$	Use $\coth^2 x = 1 + \text{cosech}^2 x$	M1
	$(3c - 1)(c - 2) = 0$ $\text{cosech}x = 2$ or $\frac{1}{3}$	Solve quadratic in cosech$x$	M1A1
	So $x = \ln\left(\frac{1 + \sqrt{5}}{2}\right)$ or $\ln\left(3 + \sqrt{10}\right)$	Use formula from part (a)	M1A1

9 **a** $\mathbf{A}\begin{pmatrix}0\\5\\2\end{pmatrix}=\begin{pmatrix}0\\20\\8\end{pmatrix}=4\times\begin{pmatrix}0\\5\\2\end{pmatrix}$

Multiply A times the vector and use the definition of an eigenvector: $\mathbf{Ax} = \lambda\mathbf{x}$

M1

So the vector is an eigenvector with eigenvalue of 4

A1

b $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=-3\begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow$

Again use the basic definition of an eigenvector

M1

$3x = -3x$ and so $x = 0$

B1

$kx + 2y + 5z = -3y$

$lx + 2y - z = -3z$

Solving: $y = -z$

Solve to find a solution

M1

So the eigenvector is: $p\begin{pmatrix}0\\1\\-1\end{pmatrix}$, for some constant p

A1

c $\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix}=\lambda\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow$

$3 = \lambda$ and so the eigen value is 3

M1A1

$k + 7 = 3$ so $k = -4$

$l + 1 = 3$ so $l = 2$

M1A1A1