ADVANCED MATHEMATICS
EXTENSION PAPER
Raising standards in mathematics education

ADVANCED MATHEMATICS EXTENSION PAPER

April 2016

Paper 4

Time Allowed: 3 hours
You may use a calculator.

Information for Candidates:

- Write your answers in black ink or ball-point pen.
- This paper is divided into **four** sections. The instructions at the start of each section will tell you how many questions to answer.
- You may use the *Mathematical Formulae and Statistical Tables* booklet.
- The maximum mark for this paper is 200, including 5 marks for style, clarity, and presentation, and 5 marks for technique and versatility (scored based on the number of questions to which complete and correct solutions are produced).

Advice to Candidates:

- You must ensure that your answers to parts of questions are clearly labelled.
- You must show sufficient working in each question to make clear the **mathematical** method used.
- Non-exact answers should be given to 3 significant figures, or 1 decimal place for answers in degrees.
- You are reminded of the necessity for good English and orderly presentation in your answers.

FOR EXAMINER'S USE ONLY:

Question	A1	Ac1	Ac2	Ac3	B6	B7	B8	Cc1	Cc2	Dc1	Dc2	Dc3	Dc4
Maximum Mark	20	20	20	20	18	19	13	10	10	15	15	15	15
Mark Awarded													

Question	Style	Technique	TOTAL
Maximum Mark	5	5	200
Mark Awarded			

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Section A: 60 marks

Answer **Question 1** and **two** other questions. Each question is worth 20 marks.

1. Consider the function

$$f(x) = \frac{e^x}{3 + e^x}$$

Show that $f(x)$ has no stationary points, and find the coordinates of its point of inflection. **(7)**

Show also that $0 < f(x) < 1$ for all x, and hence sketch the curve $y = f(x)$, justifying its behaviour as $x \to \infty$ and as $x \to -\infty$. **(7)**

Explain why $f(x)$ has an inverse, and find $f^{-1}(x)$ in terms of x. **(6)**

2. Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{1 + \tan^2 x}{1 + \tan x}\,\mathrm{d}x$$

(6)

and

$$\int_1^{\ln 3} \frac{1 + e^x}{1 - e^x}\,\mathrm{d}x$$

(7)

Show that for $n \geq 2$,

$$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{\sqrt{1 - x^n}} \leq \frac{\pi}{6}$$

(7)

3. Using the Binomial Theorem, prove the following identities:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

(3)

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$$

(4)

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}(2^{n+1} - 1)$$

(6)

$$\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n(n+1)2^{n-2}$$

(7)

SECTION A CONTINUES OVERLEAF…

4. Consider the variable point $P(2at, at^2)$ on the curve $x^2 = 4ay$.

Prove that the equation of the normal to the curve at P is $x + ty = at^3 + 2at$. **(5)**

Let Q be another point on the curve such that the normal to the curve at Q is perpendicular to the normal to the curve at P. Find the coordinates of Q. **(4)**

Show that these two normals intersect at the point R, whose coordinates are

$$x = a\left(t - \frac{1}{t}\right), y = a\left(t^2 + 1 + \frac{1}{t^2}\right)$$

(7)

Hence find the equation of the locus of R in the form $y = f(x)$. **(4)**

5. Show that if $y = e^x$, the following differential equation is satisfied:

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

(2)

Hence prove that if $y = ue^x$, where u is a function of x, then

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

(7)

By setting $v = u'(x)$ in this equation, find u in terms of x, and hence show that a solution of the original equation is $y = Ax + Be^x$, where A and B are constants. **(11)**

END OF SECTION A

TURN OVER...

Section B: 50 marks

Answer **all** the questions. The number of marks for each part-question and whole question is shown in brackets.

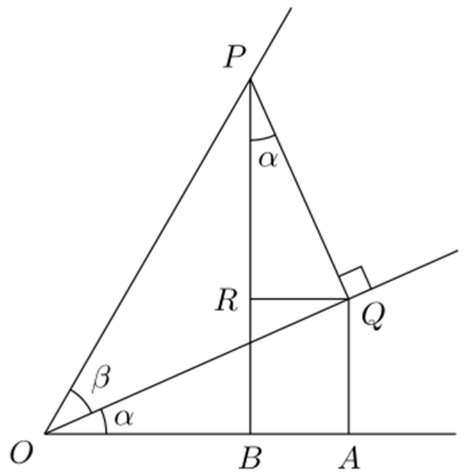
6. By considering the derivatives of $\sin^2 x$ and $\cos^2 x$, show that $\sin^2 x + \cos^2 x = 1$. **(4)**
Show similarly that $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$ for all x such that $-1 \leq x \leq 1$. **(3)**

By considering suitable derivatives, prove the following identities:

$\ln x^n = n \ln x$ **(4)**
$\ln\big(f(x)g(x)\big) = \ln\big(f(x)\big) + \ln\big(g(x)\big)$ **(5)**

Deduce that these results hold regardless of the base of the logarithm, explaining your answer fully. **(2)**
[TOTAL: 18]

7.



In the diagram, $\angle QOA = \alpha$ and $\angle POQ = \beta$, and OP is of length 1. PB is perpendicular to OA and PQ is perpendicular to OQ; QR is parallel and equal in length to AB.

Write down expressions for PQ and OQ in terms of β. **(2)**

Explain why $\angle RPQ = \alpha$, and hence find expressions for AQ and PR in terms of α and β.
Hence show that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. **(6)**

Deduce that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and that

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

(4)

Hence show that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. **(5)**
[TOTAL: 19]

TURN OVER

8. Using integration by parts, show that for all non-negative integers n,

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, \mathrm{d}x$$

(8)

Hence show that

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, \mathrm{d}x = \frac{2}{3}$$

(3)

and deduce that

$$\int_0^{\frac{\pi}{2}} \sin^7 x \, \mathrm{d}x = \frac{16}{35}$$

(2)

[TOTAL: 13]

END OF SECTION B

TURN OVER...

Section C: 20 marks

Answer **two** questions. Each question is worth **10 marks**.
Aim to write about 150-200 words for each question.

9. *"Euclid's Elements was the first real mathematics textbook."* How far do you agree?

10. Is a coherent understanding of infinity possible without set theory? Explain your answer precisely.

11. Discuss the general solution of the cubic equation in the sixteenth century, with regard to both the history and the mathematics.

12. Why was the development of non-Euclidean geometry so important?

13. Give an account of the contributions of Newton and Leibniz to the development of calculus, and explain the controversy that subsequently arose.

14. What are constructivism and intuitionistic logic, and why are they controversial?

15. *"Zero: a dangerous idea"*. Discuss this statement with reference to historical examples.

16. How far do you agree with G. H. Hardy's conception of the purpose of mathematics?

17. Outline the development of modern algebraic notation up to and including the contributions of Euler.

18. Does mathematics need a philosophy?

END OF SECTION C

TURN OVER…

Section D: 60 marks

Answer **at least one** question and **not more than four** questions.
Each question is worth **15 marks**.

19. Define $[x]$ to be the *integer part* of x, e.g. $[4.7] = 4$, $[-1.8] = -1$, $[3] = 3$.
Solve the equation

$$\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \cdots + \left[\sqrt[3]{x^3 - 1}\right] = 400$$

You must show that you have found **all** the possible values of x.

20. Let P be a point inside the triangle ABC, and let L, M, and N be the feet of the perpendiculars from P to BC, AC, and AB respectively. Determine, with proof, the position of P which maximises the product $PL \times PM \times PN$.

21. Let x, y, and z be positive real numbers.
Prove that

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$$

22. A function $f(n)$ is defined on all positive integers n, and satisfies:

- $f(n) \geq 0$ for all n
- If the final digit of n is 3, then $f(n) = 0$
- $f(10) = 0$
- $f(mn) = f(m) + f(n)$ for all m, n

Prove that $f(n) = 0$ for all n.

23. Let $PQRS$ be a quadrilateral with area A, and let O be a point inside it. Prove that if $2A = OP^2 + OQ^2 + OR^2 + OS^2$, then $PQRS$ is a square and O is its centre.

24. A multiple of 17, when written in base 2, contains exactly three digits 1. Prove that it contains at least six digits 0, and that if it contains exactly seven digits 0, then it is even.

25. Find a set S of 7 consecutive positive integers for which a polynomial $P(n)$ of degree 5 exists with the following properties:

- All the coefficients of $P(n)$ are integers
- $P(n) = n$ for 5 of the 7 elements of S, **including the greatest and least elements**
- $P(n) = 0$ for 1 of the 7 elements of S

26. Let a_n be the number of values of $\binom{n}{r}$ (where $0 \leq r \leq n$) which leave remainder 1 on division by 3, and let b_n be the number which leave remainder 2 instead. Prove that $a_n > b_n$ for all positive integers n.

END OF QUESTIONS **TOTAL MARK = 190 (Questions) + 10 (Style & Technique) = 200**