Summer 2013 examination

MA301
Game Theory I

Half Unit

Suitable for all candidates

Instructions to candidates

Time allowed: 2 hours

This paper contains **5** questions. You may attempt as many questions as you wish, but only your **BEST 4** answers will count towards the final mark. All questions carry equal numbers of marks.

Please write your answers in dark ink (preferably black or blue) only.

Answers should be justified by showing work.

You are supplied with: Mathematics Answer Booklets

Calculators are **not** allowed in this examination.

1

Recall that in the game Nim, the two players start with some heaps of chips and alternately remove some chips from one of these heaps. Player I moves first. The player to remove the last chip wins.

(a) Consider a Nim game with four heaps of sizes 2, 3, 5 and 7. Find all winning moves from this position, if any. You may use the known way of determining a winning strategy in Nim.

(b) In a Nim game with four heaps of chips, what is the maximum number of different winning moves? Explain.

The following questions refer to a new game called SNATCH, which is played as follows. As in Nim, there are heaps of chips, and a possible move is to remove some number of chips from a single heap. In addition, whenever there are among the heaps two heaps of *equal* size, then an extra allowed move is to remove *both* heaps completely in one go. As before, the last player who can make a move wins.

(c) Explain why the extra rule in SNATCH makes this game different from ordinary Nim, by describing a game situation that is winning in SNATCH but not in Nim and vice versa.

(d) Show, using the mex rule, that the Nim value of a single heap of chips in SNATCH is the same as its size.

(e) Explain why for two heaps of chips, the Nim value in SNATCH is not the Nim-sum of the Nim values of the individual heaps.

(f) Find the Nim values of the following situations in SNATCH: Two heaps of sizes 3 and 1; and three heaps of sizes 3, 1, and 1.

(g) Consider a SNATCH game with n heaps, all of size 1. For which n is this a winning situation (for the first player to move) and for which n a losing situation?

2

Consider the following bimatrix game.

I \ II	a	b	c	d
T	4	3	2	0
	1	4	0	1
B	1	4	5	6
	2	2	3	0

(a) Find all Nash equilibria of the game in pure or mixed strategies, and their payoffs to the two players.

(b) Write down the game tree for the commitment game derived from the above game where player I moves first, and player II is informed about the move of player I and can react separately with any of her choices, with the payoffs to the players as above.

(c) Find all subgame perfect equilibria of the game in (b). Find a Nash equilibrium in pure strategies that is not subgame perfect which has different payoffs than any subgame perfect equilibrium.

(d) Consider the commitment game for the game above where player I can commit to a *mixed* strategy x. Then player II learns x, but not the resulting pure strategy that is chosen according to x, and chooses her best response to x. Show that in this game, player I can get a better payoff than in any of the Nash equilibria in (a) and the pure-strategy subgame perfect equilibria in (c), by giving one such mixed strategy x of player I.

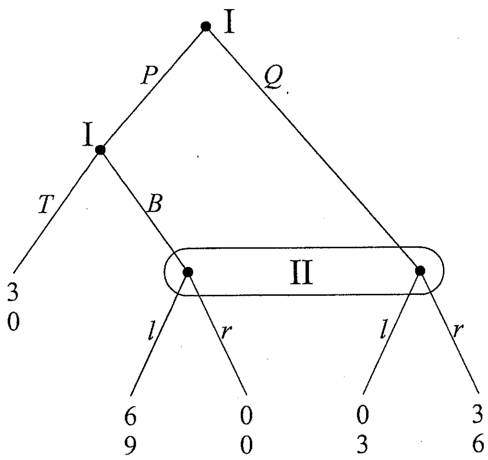
(e) Repeat (b), (c), (d) for the game where player II commits by moving first and player I moves second (where in the game corresponding to (d), player II commits to a mixed strategy y).

3

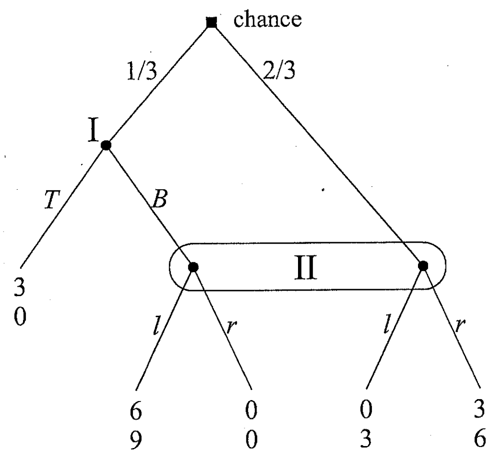
(a) For a game in extensive form with information sets, define what it means to say that a player has perfect recall.

Determine which of the following games has perfect recall. For the games that do have perfect recall, find all their Nash equilibria in mixed or pure strategies, find all proper subgames of the respective game, and state which equilibria are subgame perfect. Justify your answers.

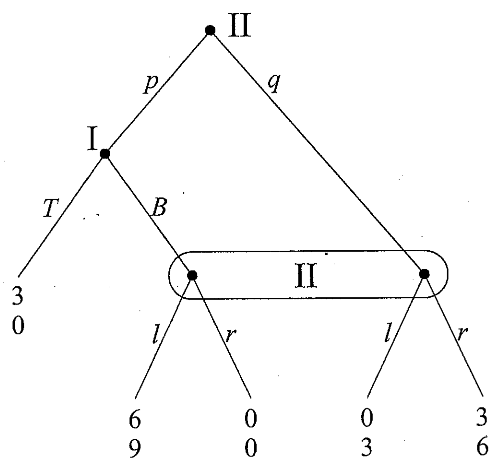
(b)



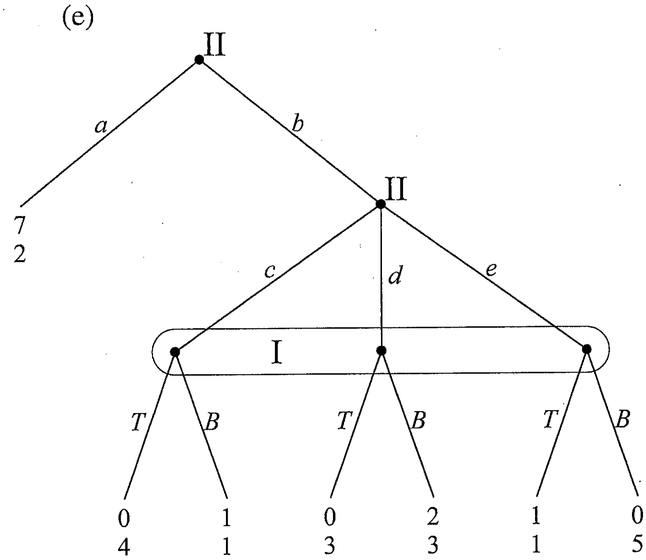
(c)



(d)



(e)



4

Consider the following bargaining problem. In the usual way, a unit "pie" is split into nonnegative amounts x and y with $x+y \leq 1$. The utility function of player I is given by $u(x) = x^{\alpha}$, for some α so that $0 < \alpha < 1$, and the utility function of player II by $v(y) = y$. The disagreement point is $(0,0)$.

(a) Show that both utility functions u and v are increasing and concave.

(b) Find the Nash bargaining solution of this problem. Write down how the *pie* is split into x and y for player I and II, respectively. Show that $0 < x < 1/2$.

Assume now that the bargaining outcome is determined by the subgame perfect equilibrium of the standard alternating-offers bargaining game with k rounds. In round 1 player I makes a certain demand. If player II rejects this, she can make a certain counterdemand in round 2, and so on. A rejection in the last round k means that both players get nothing. The last player to make a demand is player I if k is odd, and player II if k is even. When agreement takes place in round i for some i with $1 \leq i \leq k$, then the bargaining set has shrunk by a factor of δ^{i-1}. That is, the utility of what the players can get after each round of rejection is reduced by multiplication with the discount factor δ, where $0 < \delta < 1$.

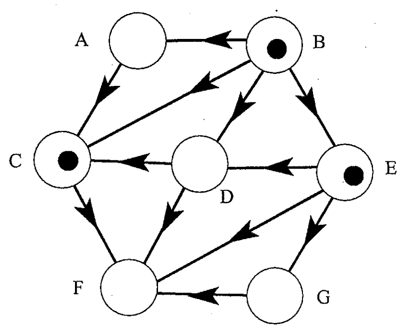
(c) Consider the case when $k = 2$, and so the game has two rounds. In this case, find the subgame perfect equilibrium demand x of player I in round 1, for general δ.

(d) Consider the case when $k = 3$, and so the game has three rounds. In this case, find the subgame perfect equilibrium demands x of player I in round 1, and y of player II in round 2, for general δ.

(e) Consider now an infinite number of rounds. Determine stationary equilibrium strategies of the game, where player I demands the same amount x each time, and player II demands the same amount y each time. For such strategies, find the demands x and y when δ goes to 1, and compare your answer with that found in part (b).

5

Consider the following directed graph. Each node is marked with one of the letters A to G. The nodes B, C and E each have a counter on them. In any move, one of the counters is moved to a neighbouring node in the direction of the arrow, for example from B to C (so counters are allowed to share a node), but not from C to B.
This game is played as a *game-sum* with a single Nim heap of 7 chips, shown on the right, so a player can either move on the graph or remove some chips from the Nim heap. Players alternate, and the last player able to move wins.

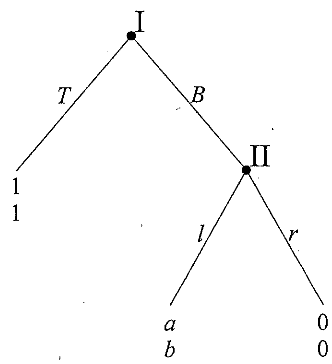


$*7$

(a) Who is winning in the above game-sum? If it is player I (the first player to move), describe *all* possible winning moves (first move only). Justify your answer.

(b) Consider now a game where the three counters on the graph may be placed differently, and the Nim heap on the right contains n instead of 7 chips. What is the smallest n so that this is *always* a winning position, irrespective of the distribution of counters on the left? Justify your answer.

The remaining questions refer to the following game, where a and b are positive real numbers.



For each of the following statements, state if they are true or false, and justify each answer with a brief argument; you will *not* get credit for just saying "true" or "false".

(c) The game has always (for any $a, b > 0$) at least two different Nash equilibria.

(d) The game has always a Nash equilibrium where player II mixes both actions.

(e) The game is always degenerate.

(f) For some $a, b > 0$, there is a Nash equilibrium where player II's payoff is neither 1 nor b.

(g) In every subgame perfect equilibrium, the payoff to player II is always at least b.