Integration C4 Questions

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Total Marks: 175 (for compulsory questions)

1. (a) Show clearly that

$$\int 2^x dx = \frac{2^x}{ln(2)} + C$$

where C is an arbitrary constant

(2)

Hence or otherwise, find

(b)

$$\int x2^x dx$$

(3)

Now suppose the following integral:

$$I_n = \int_0^1 x^n 2^x dx$$

(c) **OPTIONAL** By using integration by parts or otherwise, Show that

$$ln(2)I_n = 2 - nI_{n-1}$$

where $n \geq 2$.

(6)

(d) Hence or otherwise, explain carefully why $I_{n+1} < I_n$, state the value of n for the maximum value of I_n for $n \geq 2$. Find this maximum value of I_n.

(6)

2. The figure below shows a sketch of the area between $y = \sqrt{x - x^2}$ and the x-axis, where $0 \leq x \leq 1$.

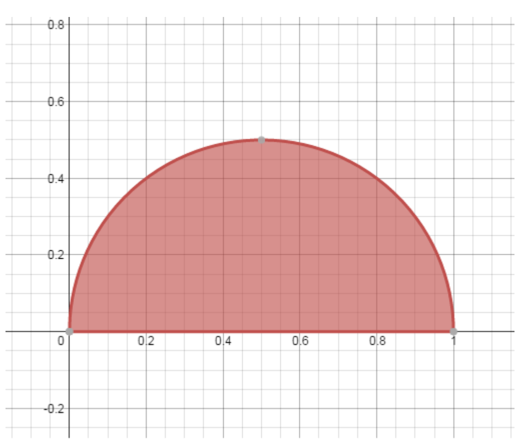

Figure 1

(a) Find the cartesian equation of this function and show the area under the function in the above figure and between the x-axis is $\frac{\pi}{8}$

(3)

Let

$$I = \int_0^1 (\sqrt{x}\sqrt{1-x})dx$$

(b) By using the trapezium rule with height of $\frac{1}{4}$, approximate the value of I to 2 decimal places.

(2)

(c) By using the substitution $x = \sin^2\theta$ or otherwise, Show that

$$I = \frac{\pi}{8}$$

(6)

3. (a) By use of an appropriate substitution, show that

$$\int \cot(x)dx = ln(\sin(x)) + C_1$$

$$\int \tan(x)dx = ln(\sec(x)) + C_2$$

Where C_1 and C_2 are arbitrary constants.

(3)

(b) Hence solve the two integrals

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sqrt{cosec^2(2x) - 1})dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sqrt{\sec^2(4x) - 1})dx$$

(5)

(c) **OPTIONAL** Show that for $n > 0$

$$J_1 = \int_0^{\frac{\pi}{4}} \tan^n(x)\sec^2(x)dx = \frac{1}{n+1}$$

and show that $J_1 = J_2$ where

$$J_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n(x)cosec^2(x)dx$$

[**NOTE**: You may assume that as $x \to \frac{\pi}{2}$ then $\cot(x) \to 0$ i.e at $x = \frac{\pi}{2}, \cot(x) = 0$]

(9)

4. Given that $f(x) = \sqrt{4 - 7x}$

   (a) Express $f(x)$ by using the binomial expansion obtain the first four non-zero terms, simplifying your coefficients.

   (4)

   (b) By using your binomial expansion of $f(x)$ find an approximate value of A to 3 decimal places, where

$$A = \int_0^{\frac{1}{4}} f(x)dx$$

   (3)

   (c) Find the exact value of A, and show that the percentage error of our previous approximation is 0.129 %.

   (4)

5. (a) Prove the identity

$$\cos(3x) = 4\cos^3 x - 3\cos x$$

   Hence deduce an identity for $\cos 9x$ (3)

   (b) Show, using the substitution $x = \sin\theta$ or $x = \cos\theta$ that

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}dx = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

   (5)

   The figure below shows a sketch of the parametric equation valid for $0 \leq t \leq \frac{2\pi}{3}$

$$x = \sin 3t, y = \cos 9t$$

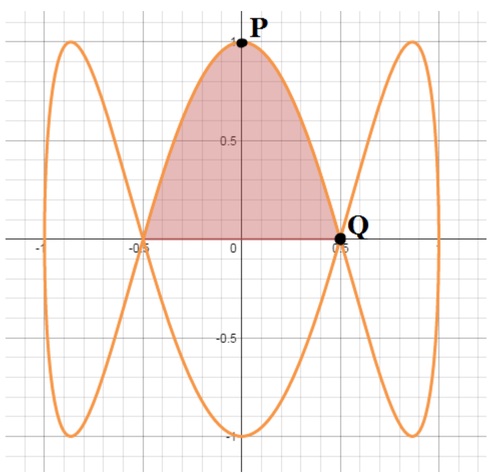

Figure 2

   (c) Given that at P, $t = 0$ and that at Q, $t = \frac{\pi}{18}$, Show that P has coordinates $(0, 1)$ and Q has coordinates $(\frac{1}{2}, 0)$.

   (1)

   The shaded region in Figure 2 is bounded by the given curve and the x-axis. We wish to find half of this shaded region (for now) between $x = 0$, $x = \frac{1}{2}$, the curve given and the x-axis

(d) Show that this initial area that we are trying to find is given by the integral

$$3\int_0^{\frac{\pi}{18}} (\cos 9t \cos 3t)dt$$

and hence find the area.

(6)

(e) By using the identity for $\cos 9x$ which your proved in part (a), show that the cartesian equation of this parametric equation is

$$y = \pm\sqrt{1-x^2}(1-4x^2)$$

(5)

Given that (which you can prove using the same substitution as part (b))

$$\int_0^{\frac{1}{2}} x^2\sqrt{1-x^2}dx = \frac{\pi}{48} - \frac{\sqrt{3}}{64}$$

(f) Show that you obtain the same answer to part (d) by integrating

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}(1-4x^2)dx$$

(4)

(g) Deduce the area of the shaded region in Figure 2.

(1)

6. (a) Solve

$$\int_1^e ln(x)dx$$

(3)

(b) Suppose the integral

$$I_n = \int_1^e x^n ln(x)dx$$

Show using calculus that

$$I_n = \frac{ne^{n+1}+1}{(n+1)^2}$$

Verify that your answer to part (a) works for the value $n = 0$. Explain carefully why this result is not valid for $n = -1$, and hence find I_{-1} separately.

(9)

(c) Explain whether I_{n+1} is greater than, less than or equal to I_n using the given formula.

(2)

(d) By using a substitution of $u = \sin x$ solve

$$\int ln(\sin(x))\cot(x)dx$$

(5)

(e) **OPTIONAL** By using a substitution $u = \pi - x$ Show that,

$$\int_0^{\pi} \frac{ln(\sin(x))}{\cos(x)} dx = 0$$

(5)

7. (a) Show using partial fractions that

$$\frac{4x}{x^4 - 1} \equiv \frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}$$

(4)

(b) Show by separation of variables that the differential equation

$$\frac{dy}{dx} = \frac{\sin^4 y - 1}{2x \sin 2y}$$

becomes

$$\int \frac{4 \sin y \cos y}{\sin^4 y - 1} dy = \int \frac{1}{x} dx$$

(1)

(c) By using a substitution $u = \sin y$ and given that at $x = 4$ $y = 0$, solve the differential equation and show that

$$x = \frac{4\left|\sin^2 y - 1\right|}{\sin^2 y + 1} \equiv \frac{4(1 - \sin^2 y)}{\sin^2 y + 1}$$

Explain the need for the modulus or change of sign of the function of x.

(10)

(d) Hence find the value of x when $y = \frac{\pi}{6}$

(1)

8. (a) Given that

$$y = \frac{ax + b}{cx + d}$$

Find $\frac{dy}{dx}$ in its simplest form.

(2)

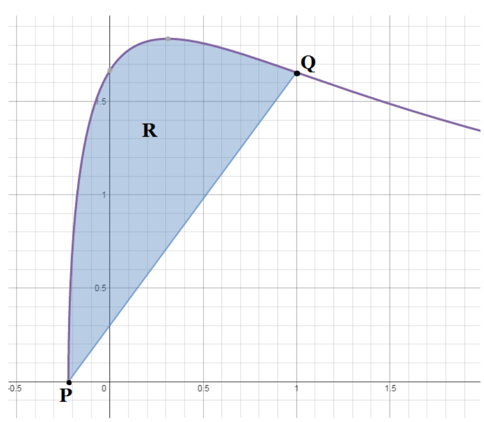

Figure 3

(b) The diagram above shows a sketch of

$$f(x) = \frac{4}{x+2}\sqrt{\ln\left(\frac{2(5x+2)}{x+2}\right)}$$

with the two labelled points, P and Q, and the area R which is bounded by the given function and a linear function l, that passes through both points P and Q. Find the exact coordinates of P and Q where the x-coordinate at Q is 1, and P is the x-intercept of $f(x)$. Explain carefully why $f(x) \geq 0$ for values of x greater than or equal to the x-coordinate of P, and explain why all other values of x do not give real values of $f(x)$.

(5)

(c) The function $f(x)$ is rotated 2π radians about the x-axis. Find the volume generated by $f(x)$ between the x-coordinates of P and Q, giving your answer in exact form.
[**HINT**: part (a) will be useful to notice the substitution for this integral.]

(8)

(d) The linear function l is also rotated 2π radians about the x-axis. Given that the shape it makes is a cone, and the volume of a cone is given by $V = \dfrac{\pi r^2 h}{3}$ where r is the radius of the cone and h is the height of the cone. Find the exact volume of the cone that l makes when rotated 2π radians.

(2)

(e) Deduce the volume that R generates when it is rotated 2π radians about the x-axis, giving your answer to 2 decimal places.

(2)

END