STEP II Q11

(i) Let A, B denote the particles with positions $\mathbf{r}^A = (a + ut\cos\alpha, ut\sin\alpha)$ and $\mathbf{r}^B = (vt\cos\beta, b + vt\sin\beta)$ respectively. Rearranging to eliminate t and writing the trig functions in harmonic form, the particles collide iff

$$\begin{aligned}
\mathbf{r}^A(t) = \mathbf{r}^B(t) \quad &\iff \quad \frac{a}{v\cos\beta - u\cos\alpha} = \frac{b}{u\sin\alpha - v\sin\beta} \\
&\iff \quad u(a\sin\alpha + b\cos\alpha) = v(a\sin\beta + b\cos\beta) \\
&\iff \quad u\sqrt{a^2+b^2}\sin(\theta+\alpha) = v\sqrt{a^2+b^2}\sin(\theta+\beta) \\
&\iff \quad \boxed{u\sin(\theta+\alpha) = v\sin(\theta+\beta)}
\end{aligned}$$

where θ satisfies $\tan\theta = \dfrac{b}{a}$, as desired.

(ii) Observe that the positions of the projectile and the gun bullet w.r.t. the foot of the gun tower are given by $\mathbf{r}^P = (a + ut\cos\alpha, ut\sin\alpha - gt^2/2)$ and $\mathbf{r}^G = (vt\cos\beta, b + vt\sin\beta - gt^2/2)$ respectively. Let T denote the time of collision after projection. Considering the height at which the collision occurs:

$$\mathbf{r}^P_y = \mathbf{r}^G_y \iff T = \frac{b}{u\sin\alpha - v\sin\beta}$$

Furthermore, observe that P hits the ground again at time $T_0 > 0$, where T_0 satisfies:

$$\mathbf{r}^P_y(T_0) = 0 \iff T_0 = \frac{2u\sin\alpha}{g}$$

Given that the impact occurs before P reaches the ground, it follows that:

$$T_0 > T \iff \boxed{2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg}$$

as required.
Now, we see that the bullet hits the projectile if and only if A and B collide. Indeed:

$$\mathbf{r}^P = \mathbf{r}^G \iff \mathbf{r}^P + (0, gt^2/2) = \mathbf{r}^G + (0, gt^2/2) \iff \mathbf{r}^A = \mathbf{r}^B$$

Given that the cartesian trajectories of A and B are given by the straight lines $y = \tan\alpha x - a\tan\alpha$ and $y = x\tan\beta + b$ respectively, in order for there to be an intersection and therefore a collision, it is necessary that the gradient of the former line is more steep than the latter (as seen from a quick sketch). That is, G hits P only if

$$\tan\alpha > \tan\beta \implies \boxed{\alpha > \beta}$$

As the angles are acute.