Let

$$I = \int\limits_0^{\frac{\pi}{4}} \frac{x}{\cos x \, (\cos x + \sin x)} \, \mathrm{d}x$$

Apply $(\star)$. Then we have

$$I = \frac{\pi}{8} \int_0^{\frac{\pi}{4}} \frac{1}{\cos x \, (\cos x + \sin x)} \, \mathrm{d}x$$

Now let $x = \arctan t$, so

$$I = \frac{\pi}{8} \int_0^1 \frac{1}{1+t} \, \mathrm{d}t = \frac{\pi}{8} \ln 2$$

as required.