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Consider the equation

$z^n - 1 = 0 \Leftrightarrow z^n = 1 \Leftrightarrow z^n = e^{2k\pi i}$
$(\forall k \in \mathbb{Z})$

$\Leftrightarrow z = e^{\frac{2k\pi i}{n}}$. If we restrict k

such that $0 \leq k \leq (n-1)$, this yields n distinct complex roots to the equation $z^n - 1 = 0$, the "n^{th} roots of unity":

$1 \, (=\omega^0)$, ω^k, where $k \in \mathbb{N}$, $k \leq n-1$

and $\omega = e^{\frac{2\pi i}{n}}$. And thus

$z^n - 1 = 0 \Leftrightarrow x = 1, \omega, \omega^2 \ldots \omega^{n-1}$

and hence by the fundamental theorem of algebra

$z^n - 1 = 0 \Leftrightarrow (z-1)(z-\omega)\ldots(z-\omega^{n-1}) = 0$

$\Rightarrow z^n - 1 \equiv (z-1)(z-\omega)\ldots(z-\omega^{n-1})$

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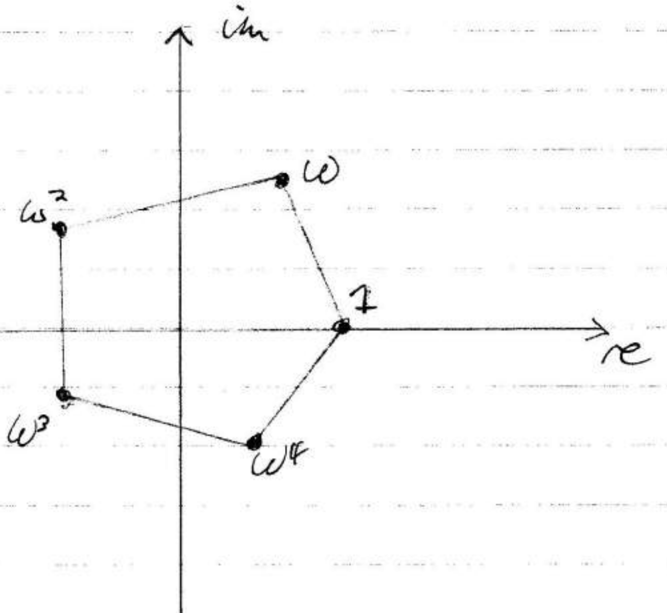
(i) Let the regular polygon $X_0 X_1 \ldots X_{n-1}$ lie in the argand diagram, and, without any loss of generality, let the vertex X_k be represented by the complex number ω^k (where $\omega = e^{\frac{2k\pi i}{n}}$ as before)



An illustrative example in the case ~~$n=4$~~ $n=5$

Let P be represented in the argand diagram by complex number p

and thus $|X_0P| \times |X_1P| \ldots \times |X_{n-1}P|$

$= |1-p| \times |\omega - p| \times |\omega^2 - p| \ldots \times |\omega^{n-1} - p|$

To compute this product, we find an equation with roots $\omega^k - p$ $(k \in \mathbb{Z},\ 0 \leq k \leq n-1)$

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The equation $z^n - 1 = 0$ has roots ω^k $(k \in \mathbb{Z},\ 0 \leq k \leq n-1)$ and we use a substitution $u = z - p$ to find an equation with roots $\omega^k - p$

$$u = z - p \Rightarrow \cancel{(z-p)}\ z = u + p$$

$$\Rightarrow (u+p)^n - 1 = 0 \qquad (*)$$

has roots $\omega^k - p$.

The constant term of the expansion of $(u+p)^n - 1$ is $p^n - 1$

and thus the product of the roots of $(*)$, $(\omega^{n-1} - p)(\omega^{n-2} - p)\ldots(1-p)$ must be $p^n - 1$.

$$\Rightarrow |\omega^{n-1} - p| \times |\omega^{n-2} - p| \ldots \times |\omega - p| \times |1 - p|$$

$$= |p^n - 1|$$

let $p = |p|e^{i\theta} \Rightarrow p^n = |p|^n e^{ni\theta}$

$$\Rightarrow |p^n - 1| = \left| |p|^n \cos n\theta - 1 + i|p|^n \sin n\theta \right|$$

$$\Rightarrow |p^n - 1| = \sqrt{|p|^{2n}(\cos^2 n\theta + \sin^2 n\theta) - 2|p|^n \cos n\theta + 1}$$

$$\Rightarrow |p^n - 1| = \sqrt{|p|^{2n} - 2|p^n| \cos n\theta + 1}$$

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Notice that if P is equidistant from X_0 and X_1 then $\arg(p) = \frac{\pi}{n}$ or $\frac{\pi}{n} + \pi$, depending on whether p lies in the first or third quadrant respectively

$$\Rightarrow |p^n - 1| = \sqrt{|p|^{2n} - 2\cos\pi |p|^n + 1}$$

$$= |p|^n + 1$$

or $$|p^n - 1| = \sqrt{|p|^{2n} - 2\cos((n+1))\pi |p|^n + 1}$$

$$= |p|^n + 1 \text{ IF } n \text{ is even}$$

$$\text{or } |p|^n - 1 \text{ IF } n \text{ is odd}$$

and thus

$$\prod_{k=0}^{n-1} |PX_k| = \begin{cases} |OP|^n + 1 & n \text{ even} \\ |OP|^n + 1 & n \text{ odd but } P \text{ in 1st quadrant [assuming } X_0 \text{ lies at } (1,0)] \\ |OP|^n - 1 & n \text{ odd, } P \text{ in 3rd quadrant} \end{cases}$$

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(ii) Assuming the same notation as before, the product

$$|X_0X_1| \times |X_0X_2| \dots \times |X_0X_{n-1}| \quad (**)$$

is equal to $\prod_{k=1}^{n-1}(1-\omega^k)$

We now make the substitution in the equation $z^n - 1 = 0$ to find an equation with roots $1-\omega^k$ $(k \geq 0)$

let $u = 1 - z \Rightarrow z = 1 - u$

$\Rightarrow (1-u)^n - 1 = 0$ has roots $1-\omega^k$

~~the~~ Considering the expansion

$$(1-u)^n - 1 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r u^r - 1$$

$$= \sum_{r=1}^{n}\binom{n}{r}(-1)^r u^r$$

then the equation

$$\sum_{r=1}^{n}(-1)^r\binom{n}{r}u^r = 0$$ has roots ~~in~~ $1-\omega^k$.

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the product (**) is equal to
the modulus of the product of
all non-zero roots of this equation

equation, i.e the roots of

$$\frac{(1-u)^n - 1}{u} = 0$$

$$\Rightarrow \sum_{r=1}^{n} (-1)^r \binom{n}{r} u^{r-1} = 0$$

The product of the roots of
this equation is equal to the
constant term (or possibly its negative),
i.e. $\pm n$

$$\Rightarrow |x_0 x_1| \ldots \times |x_0 x_{n-1}| = |\pm n| = \underline{\underline{n}}$$

as required.