MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

FURTHER METHODS FOR ADVANCED MATHEMATICS, FP2

Practice Paper FP2-A

Additional materials: Answer booklet/paper
Graph paper
MEI Examination formulae and tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS

- Write your Name on each sheet of paper used or the front of the booklet used.
- Answer **all** the questions in Section A and **one** question from Section B.
- You **may** use a graphical calculator in this paper.

INFORMATION

- The number of marks is given in brackets [] at the end of each question or part-question.
- You are advised that you may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is **72**.

Section A (54 marks)

1 You are given the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}$.

(i) Find the characteristic equation for the matrix **M**. [2]

(ii) Hence find the eigenvalues, λ_1 and λ_2, of **M** and the associated eigen vectors. [6]

(iii) Write down a matrix **S** such that $\mathbf{S}^{-1}\mathbf{MS}$ is a diagonal matrix.

Show that $\mathbf{S}^{-1}\mathbf{MS} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. [6]

(iv) Hence find $\mathbf{M}^4$. [4]

2 **(a)** Sketch the polar curve $r = a\cos^2\theta$ for values of θ in the range $0 \le \theta < 2\pi$. [4]

(b) Series C and S are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \ldots.\binom{n}{n}\cos n\theta$$

$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \ldots.\binom{n}{n}\sin n\theta$$

(i) By considering $C + \mathrm{j}S$, show that $C + \mathrm{j}S = \left(1 + \mathrm{e}^{\mathrm{j}\theta}\right)^n$. [5]

(ii) Show that $\left(1 + \mathrm{e}^{\mathrm{j}\theta}\right) = 2\cos\frac{\theta}{2}\mathrm{e}^{\frac{\mathrm{j}\theta}{2}}$ [4]

(iii) Hence show that $C = 2^n \cos^n \frac{\theta}{2}\cos\left(\frac{n\theta}{2}\right)$ and find a similar expression for S. [5]

3 **(a)** Find the exact value of $\int_{-0.4}^{0.4} \frac{1}{25x^2 + 4}\,\mathrm{d}x$. [5]

(b) **(i)** Find $\int \frac{1}{\sqrt{4 - x^2}}\,\mathrm{d}x$. [3]

(ii) Using integration by parts, show that $\int_0^{\sqrt{3}} \arcsin\left(\frac{x}{2}\right)\mathrm{d}x = \frac{\pi}{\sqrt{3}} - 1$. [10]

Section B (18 marks)
Answer one question only

Option 1: Hyperbolic Functions

4 **(i)** By differentiating the equation $\tanh y = x$, show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{artanh} x) = \dfrac{1}{1-x^2}$. [3]

(ii) Starting with the definitions $\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$ show that

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$ [4]

(iii) Use Maclaurin's expansion to find an expansion for $\ln(1 + x)$ for terms up to and including that in x^3. [5]

(iv) Hence derive an expansion for $\operatorname{artanh} x$ for terms up to and including x^3. [3]

(v) State the domain of the function $y = \operatorname{artanh} x$ and sketch the graph of $y = \operatorname{artanh} x$. [3]

Option 2: Investigation of curves
A graphical calculator is required for this question.

5 The graphs of $x = \dfrac{t^2}{t-1}$ and $y = \dfrac{t^3}{t-1}$ are shown in Fig. 5a and Fig.5b below.

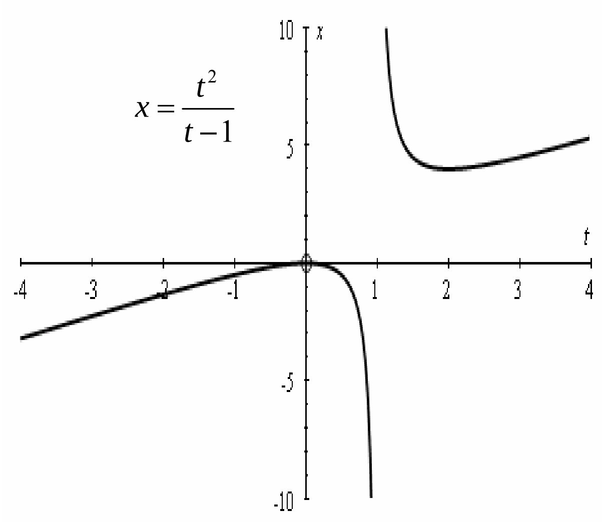

Fig. 5a

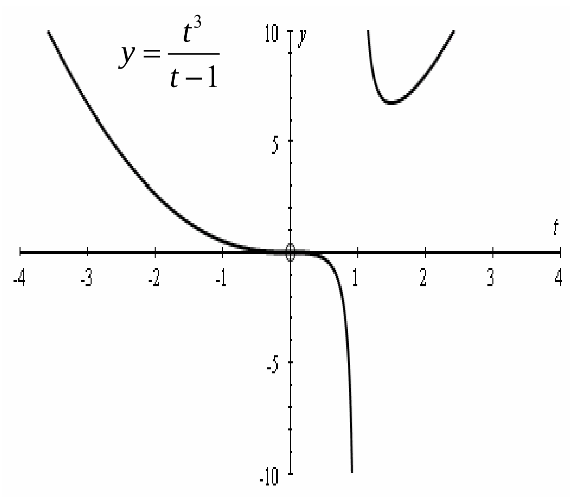

Fig. 5b

(i) Write down the values of t corresponding to the vertical asymptotes in Fig.5a and Fig. 5b. [1]

(ii) Show that the two stationary points on the graph $x = \dfrac{t^2}{t-1}$ occur at $t = 0$ and $t = 2$.

The two stationary points on the graph $y = \dfrac{t^3}{t-1}$ occur at $t = 0$ and another value of *t*. Find this value of *t*. [4]

The curve, C, with parametric equations $x = \dfrac{t^2}{t-1}$, $y = \dfrac{t^3}{t-1}$ has two distinct branches. One of these is shown in Fig. 5c below.

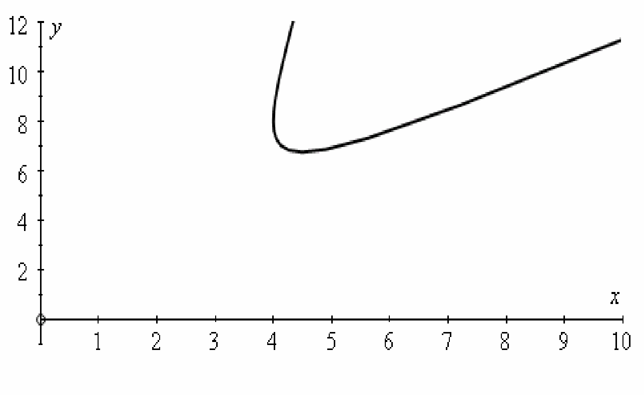

Fig. 5c

(iii) Using your graphical calculator, draw a sketch of the curve C showing both branches. Identify the point on C corresponding to $t = 0$ and describe this point. [3]

(iv) Find both the non-zero values of t **and** the cartesian coordinates of the points where the tangent to C is

(A) parallel to the y-axis,

(B) parallel to the x-axis. [6]

(v) By annotating a copy of the graph of C, clearly describe how the curve C unfolds as the parameter, t, increases from –10 to +10.Indicate increasing values of t by drawing arrows on your curve. [4]

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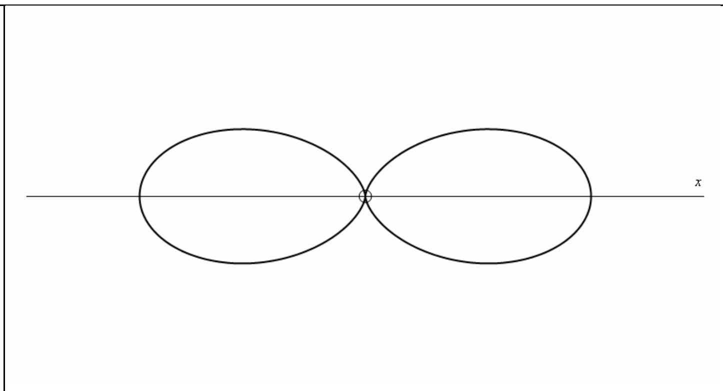
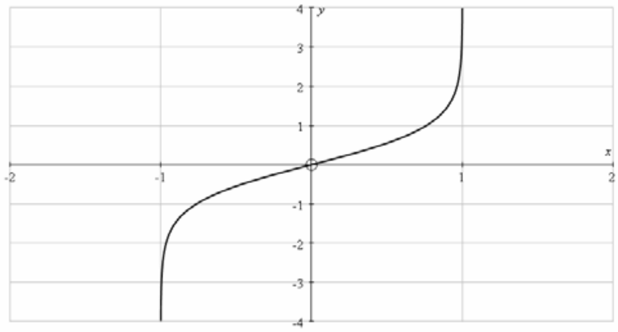
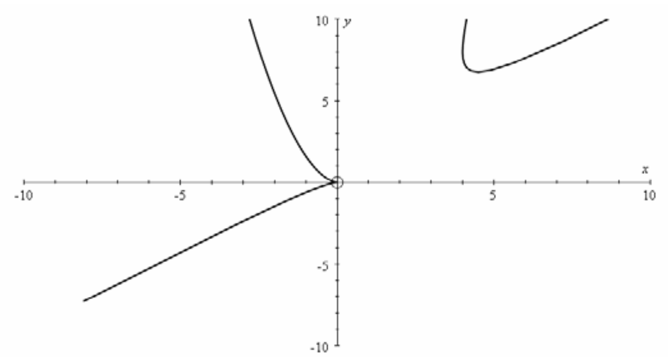
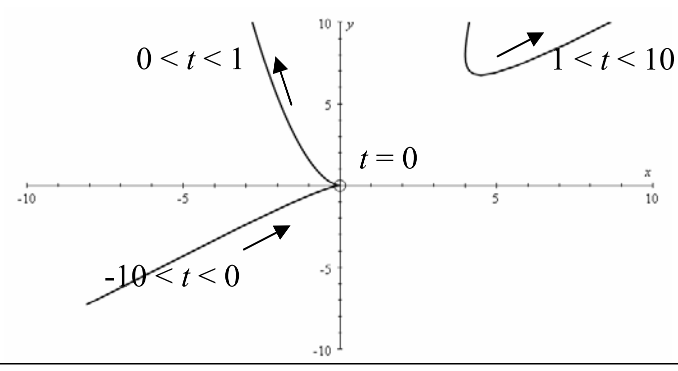
MARK SCHEME

Qu		Answer	Mark	Comment
Section A				
1	**(i)**	$\begin{vmatrix} 1-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda)-18=0$ $\Rightarrow -2+\lambda+\lambda^2-18=0 \Rightarrow \lambda^2+\lambda-20=0$	M1 A1 **2**	
	(ii)	$\lambda^2+\lambda-20=0$ $\Rightarrow (\lambda+5)(\lambda-4)=0 \Rightarrow \lambda=-5,\ 4$ For $\lambda=-5$, $\begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow x+3y=-5x$ $\Rightarrow y=-2x$ $\Rightarrow s_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ For $\lambda=4$, $\begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow x+3y=4x$ $\Rightarrow y=x$ $\Rightarrow s_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	M1 A1 M1 A1 A1 A1 **6**	
	(iii)	$S=(s_1,s_2)=\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ $S^{-1}=\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ $S^{-1}MS=\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ $=\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 \\ 10 & 4 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} -15 & 0 \\ 0 & 12 \end{pmatrix}$ $=\begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}$	B1 M1 A1 A1 M1 A1 **6**	
		$\left(S^{-1}MS\right)^4=\begin{pmatrix} -5^4 & 0 \\ 0 & 4^4 \end{pmatrix}=S^{-1}M^4S$ $\Rightarrow M^4=\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5^4 & 0 \\ 0 & 4^4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ $=\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5^4 & -5^4 \\ 2.4^4 & 4^4 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 5^4+2.4^4 & -5^4+4^4 \\ -2.5^4+2.4^4 & 2.5^4+4^4 \end{pmatrix}$ $=\frac{1}{3}\begin{pmatrix} 1137 & -369 \\ -738 & 1506 \end{pmatrix}=\begin{pmatrix} 379 & -123 \\ -246 & 502 \end{pmatrix}$	M1 A1 A1 A1 **4**	

2	**(a)**		B1 B1 B1 B1 **4**	For $0 < \theta < 90$ For $90 < \theta < 180$ For $180 < \theta < 270$ For $270 < \theta < 360$
	(b) (i)	$C + \mathrm{j}S = 1 + \binom{n}{1}(\cos\theta + \mathrm{j}\sin\theta) + \binom{n}{2}(\cos 2\theta + \mathrm{j}\sin 2\theta) + \ldots.$ $= 1 + \binom{n}{1}\mathrm{e}^{\mathrm{j}\theta} + \binom{n}{2}\mathrm{e}^{2\mathrm{j}\theta} + \ldots.. + \binom{n}{n}\mathrm{e}^{n\mathrm{j}\theta}$ $= \left(1 + \mathrm{e}^{\mathrm{j}\theta}\right)^n$	B1 M1 A1 A1 A1 **5**	
	(ii)	$1 + \mathrm{e}^{\mathrm{j}\theta} = 1 + \cos\theta + \mathrm{j}\sin\theta = 2\cos^2\frac{\theta}{2} + 2\mathrm{j}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \mathrm{j}\sin\frac{\theta}{2}\right) = 2\cos\frac{\theta}{2}\mathrm{e}^{\frac{\mathrm{j}\theta}{2}}$	M1 A1 M1 A1 **4**	
	(iii)	$C + \mathrm{j}S = \left(1 + \mathrm{e}^{\mathrm{j}\theta}\right)^n = \left(2\cos\frac{\theta}{2}.\mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}\right)^n = 2^n\cos^n\frac{\theta}{2}\mathrm{e}^{\frac{\mathrm{j}n\theta}{2}}$ $= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + \sin\frac{n\theta}{2}\right)$. Equating real parts $\Rightarrow C = 2^n\cos^n\frac{\theta}{2}.\cos\frac{n\theta}{2}$ Equating imaginary parts $\Rightarrow S = 2^n\cos^n\frac{\theta}{2}.\sin\frac{n\theta}{2}$	M1 A1 A1 M1 A1 **5**	

3	**(a)**	$\int_{-0.4}^{0.4} \frac{1}{25x^2+4}\,\mathrm{d}x = \left[\frac{1}{10}\arctan\left(\frac{5x}{2}\right)\right]_{-0.4}^{0.4}$ $= \frac{1}{10}\left(\arctan 1 - \arctan(-1)\right)$ $= \frac{1}{10}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{20}$	M1 A1 A1 M1 A1 **5**	 For 1/10 For $5x/2$
	(b) (i)	$\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{2}\right) + c$	M1 A1 A1 **3**	Style of integral Arcsin Divide by 2
	(b) (ii)	$I = \int_0^{\sqrt{3}} \arcsin\left(\frac{x}{2}\right)\,\mathrm{d}x$ $u = \arcsin\left(\frac{x}{2}\right) \quad \mathrm{d}v = \mathrm{d}x$ $\Rightarrow \mathrm{d}u = \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x \quad v = x$ $= \left[x\arcsin\left(\frac{x}{2}\right)\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x$ $= \left(\sqrt{3}\frac{\pi}{3} - 0\right) - \left[-\sqrt{4-x^2}\right]_0^{\sqrt{3}}$ $= \sqrt{3}\frac{\pi}{3} - \left(-\sqrt{1} - (-2)\right) = \sqrt{3}\frac{\pi}{3} - 1$	M1 A1 A1 A1 A1 A1 M1 A1 A1 A1 **10**	

4	**(i)**	$\tanh y = x \Rightarrow \operatorname{sech}^2 y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1-\tanh^2 y} = \dfrac{1}{1-x^2}$	M1 A1 A1 **3**	Diffn implicitly
	(ii)	$\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right),\ \cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$ $\tanh x = \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} = \dfrac{\mathrm{e}^{2x}-1}{\mathrm{e}^{2x}+1}$ Let $y = \operatorname{artanh}x$; then $x = \tanh y = \dfrac{\mathrm{e}^{2y}-1}{\mathrm{e}^{2y}+1}$ $\Rightarrow \left(\mathrm{e}^{2y}+1\right)x = \mathrm{e}^{2y}-1$ $\Rightarrow \mathrm{e}^{2y}\left(x-1\right) = -1-x$ $\Rightarrow \mathrm{e}^{2y} = \dfrac{1+x}{1-x} \Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right)$ $\Rightarrow \operatorname{artanh}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	M1 A1 M1 A1 **4**	Change subject and take logs
	(iii)	$\mathrm{f}(x) = \ln(1+x) \Rightarrow \mathrm{f}(0) = 0$ $\mathrm{f}'(x) = \dfrac{1}{1+x} \Rightarrow \mathrm{f}'(0) = 1$ $\mathrm{f}''(x) = -1(1+x)^{-2} \Rightarrow \mathrm{f}''(0) = -1$ $\mathrm{f}'''(x) = 2(1+x)^{-3} \Rightarrow \mathrm{f}'(0) = 2$ $\Rightarrow y = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - .....$	M1 A1 A1 A1 A1 **5**	
	(iv)	$\operatorname{artanh}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}\left(\ln\left(1+x\right) - \ln\left(1-x\right)\right)$ $= \dfrac{1}{2}\left(\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3}\right) - \left(-x - \dfrac{x^2}{2} - \dfrac{x^3}{3}\right)\right)$ $= \dfrac{1}{2}\left(2x + \dfrac{2x^3}{3}\right) = x + \dfrac{x^3}{3} + ...$	M1 B1 A1 **3**	Ability to write an expansion for $\ln(1-x)$ from their $\ln(1+x)$
	(v)	Domain is $\lvert x \rvert < 1$ 	B1 B1 B1 **3**	Domain General shape Asymptotes clear

5	**(i)**	$t = 1$	B1 **1**	
	(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(t-1)2t - t^2}{(t-1)^2} = \frac{t^2 - 2t}{(t-1)^2}$ $= 0$ when $t^2 - 2t = 0 \Rightarrow t = 0, 2$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(t-1)3t^2 - t^3}{(t-1)^2} = \frac{2t^3 - 3t^2}{(t-1)^2}$ $= 0$ when $2t^3 - 3t^2 = 0 \Rightarrow t = 0\,(\text{twice})$ and $t = \frac{3}{2}$	M1 A1 M1 A1 **4**	
	(iii)	 The origin is a cusp	B2 B1 **3**	Graph
	(iv)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t(2t-3)}{t-2}$ (A) When $t = 2$, the point is (4, 8) (B) When $t = 1.5$, the point is (4.5, 6.75)	M1 A1 B1 B1 B1 B1 **6**	
	(v)		B1 B1 B1 B1 **4**	Branches -10 < t < 0 0 < t < 1 1 < t < 10 All arrows correct