QUESTION 7

$\omega = e^{\frac{2\pi i}{n}} \Rightarrow 1, \omega, \omega^2, ..., \omega^{n-1}$ are the n^{th} roots of 1

Consider the product $P(z) = (z-1)(z-w)(z-\omega^2)...(z-\omega^{n-1})$

Putting $z = 1, z = \omega, z = \omega^2, ..., z = \omega^{n-1}$ into $P(z)$ successively we get zero each time.

Therefore $P(z)$ is a polynomial in z such that, when $P(z)=0$ its roots are all the n^{th} roots of 1.

Therefore we can write $P(z) = z^n - 1$.

Part (i)

Let $X_k = e^{\frac{2k\pi i}{n}}$ for $k = 0, 1, 2, ..., n-1$

Let point $P = ae^{\frac{\pi i}{n}}$ for $a \in \mathbb{R}, a \neq 0$

Putting $z = P$ into $P(z)$ we have :-

$$(P - X_0)(P - X_1)...(P - X_{n-1}) = P^n - 1$$

Since $|P - X_k| = |PX_k|$ (as defined) we have:-

$$|PX_0| \times |PX_1| \times |PX_2| \times ... \times |PX_{n-1}| = \left| \left(ae^{\frac{\pi i}{n}} \right)^n - 1 \right|$$

$$= |-a^n - 1| = a^n + 1 = |OP|^n + 1$$

(Note: for all $n \in \mathbb{N}$ $\left(e^{\frac{\pi i}{n}}\right)^n = -1$ and n even $\Rightarrow a^n > 0$ for all $a \neq 0$)

Suppose n is odd.

If $a > 0$ with n odd $\Rightarrow a^n > 0$ and we have the same result as above.

If $a < 0$ with n odd $\Rightarrow a^n < 0 \Rightarrow \left(ae^{\frac{i\pi}{n}}\right)^n = |a|^n > 0$ then the result becomes as follows:-

$$|PX_0| \times |PX_1| \times |PX_2| \times \ldots \times |PX_{n-1}| = \left|\left(ae^{\frac{\pi i}{n}}\right)^n - 1\right|$$

$$= \left|-a^n - 1\right| = \left||a|^n - 1\right| = \begin{cases} |OP|^n - 1 \text{ for } |a| \geq 1 \\ 1 - |OP|^n \text{ for } |a| < 1 \end{cases}$$

Part (ii)

Consider the equation $\sum_{k=0}^{n-1} z^k = 0$

Now $(z-1)\sum_{k=0}^{n-1} z^k = z^n - 1 = 0$

We know the roots of this equation, they are the n^{th} roots of 1. By multiplying by $z-1$ we have simply included the root 1.

Therefore $\sum_{k=0}^{n-1} z^k = 0$ must have all the n^{th} roots of 1 except 1.

Therefore $\sum_{k=0}^{n-1} z^k = (z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$

Putting $z = 1$ we have :-

$$\Rightarrow n = (1-\omega)(1-\omega^2)\dots(1-\omega^{n-1})$$

Taking the modulus of both sides:-

$$n = |X_0X_1| \times |X_0X_2| \times \dots \times |X_0X_{n-1}|$$