Theorem For $a,b,n\in\mathbb{Z}$, $a^2+b^2/(\mathrm{ab}-1)=n\Rightarrow n=5$ ...(1)

Proof

First, show that $a=b$ is impossible. If $a=b$ then

$2a^2/(a^2-1)=n\Rightarrow 2a^2=\mathrm{na}^2-n\Rightarrow a^2(2-n)=-n\Rightarrow a^2=-n/(2-n)$, which is clearly absurd since n and a are both positive integers.

Now, suppose we have a solution for the equation in (1). Without loss of generality, let a>b.

$a^2+b^2=n(\mathrm{ab}-1)$

Clearly then, a must divide b^2+n. Hence, we write $b^2+n=\mathrm{ak}$ for some $k\in\mathbb{Z}^+$. Therefore,

$k=(b^2+n)/a=\mathrm{nb}-a$ ...(2)

$\Rightarrow a=\mathrm{nb}-k$ ...(3)

Multiply by k,

$\mathrm{ak}=\mathrm{nbk}-k^2$ ...(4)

From (2), recall that $k=(b^2+n)/a$. Substituting in (4),

$b^2+n=\mathrm{nbk}-k^2$

$\Rightarrow b^2+k^2=n(\mathrm{bk}-1)$ ...(6)

$\Rightarrow$ If (a,b) is a solution, then (b,k) is a solution too. Recall that $k\in\mathbb{Z}^+$.

- If $k>b$, then recall from (3) that

$a=\mathrm{nb}-k<\mathrm{nb}-b$

$\Rightarrow a<b(n-1)$

Multiplying by a,

$\Rightarrow a^2<\mathrm{ab}(n-1)$

Adding b^2

$\Rightarrow a^2+b^2<\mathrm{ab}(n-1)+b^2$

Adding n,

$a^2+b^2+n<\mathrm{ab}(n-1)+b^2+n$

Dividing by n,

$(a^2+b^2+n)/n<(\mathrm{abn}-\mathrm{ab}+b^2+n)/n$ ...(7)

Recall from (1) that $\mathrm{nab}-n=a^2+b^2$. Therefore, $(a^2+b^2+n)/n=\mathrm{ab}$. Substituting in (7) and simplifying the RHS,

$\mathrm{ab}<\mathrm{ab}-\mathrm{ab}/n+b^2/n+1$

$\Rightarrow \mathrm{ab}<b^2+n$

$\Rightarrow \mathrm{ab}-b^2<n$

$\Rightarrow \mathrm{ab}-b^2<(a^2+b^2)/(\mathrm{ab}-1)$

Cross multiplying,

$\Rightarrow a^2b^2-\mathrm{ab}-b^3a+b^2<a^2+b^2$

$\Rightarrow a^2b^2-\mathrm{ab}-b^3a<a^2$

Dividing by a,

$\Rightarrow b^2a - b - b^3 < a$

$\Rightarrow a(b^2 - 1) < b(b^2 + 1)$

If $b \neq 1$ (there is no problem if it is, since if $b = 1, a = 2, 3$ give $n = 5$), then

$a/b < (b^2 + 1)/(b^2 - 1) < 2$

$\Rightarrow a < 2b \Rightarrow a^2 < 2\text{ab}$

Since $b^2 < \text{ab}$,

$a^2 + b^2 < 3\text{ab}$

$\Rightarrow \text{nab} - n < 3\text{ab}$

$\Rightarrow n < 3(\text{ab}/(\text{ab} - 1)) < 6, \text{since ab} \geqslant 2 \cdot 1 = 2, i.e. 2(\text{ab} - 1) > \text{ab}$

So we've proved that n<6

- If k<b, then if we have a solution for some a, b, n with $n > 5$ (WLOG a>b), choose the smallest solution pair (a, b), but then we'll have a solution (b, k) where $a > b, b > k$, so (a,b) is not the smallest pair and there is a contradiction.

It remains trivial to show that n=1,2,3,4 are impossible cases. Since n<6, this leaves n=5.

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