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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

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අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2009 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர(உயர் தர)ப் பரீட்சை, 2009 ஓகஸ்ற்
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සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

* *Answer six questions only.*

1. (a) α and β are the roots of the equation $x^2+bx+c=0$, where $c \neq 0$. Find the quadratic equation, in terms of b and c, whose roots are $\alpha^3\beta^2$ and $\alpha^2\beta^3$.

   Hence, find the quadratic equation, in terms of b and c, whose roots are $\alpha^3\beta^2+\frac{1}{\alpha^2\beta^3}$ and $\alpha^2\beta^3+\frac{1}{\alpha^3\beta^2}$.

   (b) Prove that if a polynomial $f(x)$ is divided by $x-\alpha$, then the remainder is $f(\alpha)$.
   When the polynomial $f(x)$ is divided by $(x-\alpha)(x-\beta)(x-\gamma)$, where α, β and γ are unequal real numbers, the remainder takes the form $A(x-\beta)(x-\gamma)+B(x-\alpha)(x-\gamma)+C(x-\alpha)(x-\beta)$.
   Express the constants A, B and C in terms of $\alpha, \beta, \gamma, f(\alpha), f(\beta)$ and $f(\gamma)$.

   Hence, find the value of the constant k for which the remainder when x^5-kx is divided by $(x+1)(x-1)(x-2)$ contains **no** term in x.

2. (a) Find the number of different arrangements of the 10 letters which can be made from the letters of the word PHILOSOPHY.
   In how many of these arrangements do the letters H, I, S, Y appear together?
   Also find the number of different selections of 5 letters which can be made from the 10 letters of the word PHILOSOPHY.

   (b) Let $P_n=n(n+1)\cdots(n+r-1)$, where n and r are positive integers.
   Show that $nP_{n+1}=nP_n+rP_n$.

   Assuming that P_n/n is divisible by $(r-1)!$, show that $P_{n+1}-P_n$ is divisible by $r!$.
   Deduce that the product of r consecutive positive integers is divisible by $r!$.

3. (a) By using the principle of mathematical induction, prove that $(x+y)^n=\sum_{r=0}^{n}{}^nC_r x^{n-r}y^r$, where n is a positive integer, and ${}^nC_r=\frac{n!}{(n-r)!\,r!}$.

   Deduce that $(p+q)^n-p^n-q^n$ is divisible by pq, where p, q and n are positive integers.

   (b) The r^{th} term, U_r, of an infinite series is given by $\frac{(2r+1)}{(3r-2)(3r+1)}\cdot\frac{1}{7^r}$.
   Find $f(r)$ such that $U_r=f(r-1)-f(r)$.

   Hence, find $\sum_{r=1}^{n}U_r=S_n$ and the value of $\lim_{n\to\infty}S_n$.

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4. (*a*) Find the square roots of the complex number $-80-18i$ and solve the quadratic equation

$$4z^2+(16i-4)z+(65+10i)=0.$$

(*b*) Interpret the equation $\arg(z+1)=\frac{\pi}{3}$ on an Argand diagram and find the minimum value of $|z|$.

(*c*) Show that if ω is a complex root of the equation $z^3-1=0$, then ω^2 is the other complex root.

Show also that $\omega^{2k}+(1+\omega)^k=0$, where k is an odd positive integer.

Deduce that x^2+x+1 is a factor of $x^{2k}+(1+x)^k$ for odd positive integral k.

5. (*a*) **Using first principles**, find the derivative of $f(x)=\sin x$ with respect to x.

Deduce the derivative of $g(x)=\cos x$.

Differentiate

(i) $\sin\left(\ln\left(1+x^2\right)\right)$

(ii) $\cos(\sin x)$

with respect to x.

(*b*) Let $y=\sin k\theta \operatorname{cosec}\theta$ and $x=\cos\theta$, where k is a constant.

Prove that

(i) $\left(1-x^2\right)\frac{dy}{dx}-xy+k\cos k\theta=0$,

(ii) $\left(1-x^2\right)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+\left(k^2-1\right)y=0$.

(*c*) The tangent to the curve $y\left(1+x^2\right)=2$ at the point $P\left(3,\frac{1}{5}\right)$ meets the curve again at Q.

Find the coordinates of Q.

6. (*a*) Let $I_k=\int\frac{e^t}{t^k}\,dt$, where $t>0$ and k is a positive integer.

Show that $(k-1)I_k-I_{k-1}+\frac{e^t}{t^{k-1}}=C$, where C is an arbitrary constant.

Find $\int e^x\left(\frac{1-x}{1+x}\right)^2 dx$, where $x>-1$.

(*b*) f is a real valued function defined on the set of real numbers, and $J=\int_0^a f(x)\,dx$, where $a>0$.

Show that $\int_0^a f(a-x)\,dx=J$.

Evaluate $\int_0^{\pi/2}\frac{\sin^{2k}x}{\cos^{2k}x+\sin^{2k}x}\,dx$, where k is a positive integer.

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7. Show that the coordinates of any point on the straight line through the point (x_0, y_0) and perpendicular to the straight line $ax+by+c=0$ can be expressed in the form $(x_0+at,\ y_0+bt)$, where t is a parameter.

Hence, find the coordinates of the mirror image of the point (x_0, y_0) in the straight line $ax+by+c=0$.

The equations of the perpendicular bisectors of the sides OA and AB of the triangle OAB are $x\cos\theta+y\sin\theta=1$ and $x-y=1$ respectively, where $0<\theta<\frac{\pi}{2}$ and O is the origin.

Find the equations of the three sides of the triangle OAB.

Also, find the equation of the perpendicular bisector of the side OB and verify that the perpendicular bisectors of the sides of the triangle OAB are concurrent.

8. The equations $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ represent two non-intersecting circles. Let O_1 and O_2 be the centres of the two circles. A pair of common tangents can be drawn to the two circles from a point T lying between O_1 and O_2.

Identify the point T and find its coordinates in terms of the coordinates of O_1 and O_2 and the radii of the two circles.

Identify also the point T' on the extended line O_1O_2, through which a second pair of tangents can be drawn to the two circles and find its coordinates.

Find the equations of the four common tangents to the two circles $x^2+y^2-18x+6y+86=0$ and $x^2+y^2+18x-6y+74=0$.

9. (*a*) State and prove the ***sine rule*** in the usual notation.

The points A, B and C taken in the ascending order, lie on a straight line inclined at an angle θ to the horizontal. $AB=x$ and D is the point vertically above at a height h from the point C. CD subtends angles α and β at A and B respectively.

Prove that

(i) $h=\dfrac{x\sin\alpha\,\sin\beta}{\sin(\beta-\alpha)\cos\theta}$.

(ii) $d=\dfrac{x\sin(\alpha+\theta)\sin\beta}{\sin(\beta-\alpha)}$, where d is the height of D above the level of A.

(*b*) Find

(i) the general solution of the equation $\sin\theta-\cos\theta=1$,

(ii) the value of x satisfying the equation $\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}=\sin^{-1}x$.

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