Part 3 General Relativity

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1 Introduction

Special relativity has a preferred class of observers: inertial (non-accelerating) observers. Associated to any such observer is a set of coordinates (t, x, y, z) called an *inertial frame*. Different inertial frames are related by Lorentz transformations. The Principle of Relativity states that physical laws should take the same form in any inertial frame.

Newton's law of gravitation is

$$\nabla^2 \Phi = 4\pi G\rho \tag{1}$$

where Φ is the gravitational potential and ρ the mass density. Lorentz transformations mix up time and space coordinates. Hence if we transform to another inertial frame then the resulting equation would involve time derivatives. Therefore the above equation does not take the same form in every inertial frame. Newtonian gravity is incompatible with special relativity. GR is the theory that replaces both Newtonian gravity and special relativity.

2 The equivalence principle

2.1 The weak equivalence principle

The equivalence principle was an important step in the development of GR. There are several forms of the EP, which are motivated by thought experiments involving Newtonian gravity.

In Newtonian theory, one can distinguish between the notions of *inertial* mass m_I, which appears in Newton's second law: $\mathbf{F} = m_I \mathbf{a}$, and *gravitational* mass, which governs how a body interacts with a gravitational field: $\mathbf{F} = m_G \mathbf{g}$.

Note that this equation defines both m_G and $\mathbf{g}$ hence there is a scaling ambiguity $\mathbf{g} \to \lambda \mathbf{g}$ and $m_G \to \lambda^{-1} m_G$ (for all bodies). We fix this by defining $m_I/m_G = 1$ for a particular test mass, e.g., one made of platinum. Experimentally it is found that other bodies made of other materials have $m_I/m_G - 1 = \mathcal{O}(10^{-12})$ (Eötvös experiment).
The exact equality of m_I and m_G for all bodies is one form of the *weak equivalence principle*. Newtonian theory provides no explanation of this equality.
The weak EP implies that a uniform gravitational field cannot be distinguished from constant acceleration. To see why, consider a set of particles with positions $\mathbf{x}_i(t)$, inertial masses m_{Ii} and gravitational masses m_{Gi} interacting via forces that depend only on the particle separations. Assume that there is a uniform gravitational field $\mathbf{g}$. The equations of motion are:

$$m_{Ii}\ddot{\mathbf{x}}_i = m_{Gi}\mathbf{g} + \sum_{j\neq i} \mathbf{F}_{ji}(\mathbf{x}_j - \mathbf{x}_i), \tag{2}$$

where $\mathbf{F}_{ji}$ is the force of the jth particle on the ith particle.
Now consider a new frame of reference moving with constant acceleration $\mathbf{a}$ with respect to the first frame. The origin of the new frame has position $\mathbf{X}(t)$ where $\ddot{\mathbf{X}} = \mathbf{a}$. The coordinates of the new frame are $t' = t$ and $\mathbf{x}' = \mathbf{x} - \mathbf{X}(t)$. Hence the equations of motion in this frame are

$$m_{Ii}\ddot{\mathbf{x}}'_i = m_{Gi}\mathbf{g} - m_{Ii}\mathbf{a} + \sum_{j\neq i} \mathbf{F}_{ji}(\mathbf{x}'_j - \mathbf{x}'_i). \tag{3}$$

But the weak equivalence principle says that $m_{Ii}/m_{Gi} = 1$ for all i so

$$m_{Ii}\ddot{\mathbf{x}}'_i = m_{Gi}\left(\mathbf{g} - \mathbf{a}\right) + \sum_{j\neq i} \mathbf{F}_{ji}(\mathbf{x}'_j - \mathbf{x}'_i). \tag{4}$$

The laws of mechanics in the accelerating frame are the same as in the first frame but with a constant gravitational field $\mathbf{g}' = \mathbf{g} - \mathbf{a}$. If $\mathbf{g} = 0$ then the new frame appears to contain a gravitational field $\mathbf{g}' = -\mathbf{a}$: *uniform acceleration is indistinguishable from a uniform gravitational field.*
We can define an inertial frame as a reference frame in which the laws of physics take the simplest form. In the present case, it is clear that this is a frame with $\mathbf{a} = \mathbf{g}$, i.e., a freely falling frame. This gives $\mathbf{g}' = 0$ so an observer at rest in such a frame, i.e., a freely falling observer, does not observe any gravitational field. From the perspective of such an observer,

the gravitational field present in the original frame arises because this latter frame is accelerating with acceleration $-\mathbf{g}$ relative to him.
Even if the gravitational field is not uniform, it can be approximated by uniform for experiments performed in a region of space-time sufficently small that the non-uniformity is negligible. In the presence of a non-constant gravitational field, we define a *local inertial frame* to be a set of coordinates (t, x, y, z) that a freely falling observer would define in the same way as coordinates are defined in Minkowski spacetime. The word *local* emphasizes the restriction to a small region of spacetime, i.e., t, x, y, z are restricted to small values.
An alternative way of stating the weak EP is: *In a local inertial frame, particle mechanics is indistinguishable from particle mechanics in an inertial frame in Minkowski spacetime.*

2.2 The strong equivalence principle

Einstein extended the EP to encompass all of physics, not just mechanics: *In a local inertial frame, the results of all experiments will be indistinguishable from the results of the same experiments performed in an inertial frame in Minkowski spacetime.*
The experimental tests of the weak EP involve ordinary matter, composed of electrons and nuclei interacting via the electromagnetic force. Nuclei are composed of protons and neutrons, which are in turn composed of quarks and gluons, interacting via the strong nuclear force. A significant fraction of the nuclear mass arises from binding energy. The fact that this is all consistent with the weak EP is evidence that the strong EP is indeed true.
Note that we have motivated the EP by Newtonian arguments. Since we restricted to velocities much less than the speed of light, the incompatibility of Newtonian theory with special relativity is not a problem. But the EP is supposed to be more general than Newtonian theory. It is a guiding principle for the construction of a relativistic theory of gravity. In particular, any theory satisfying the EP should have some notion of "local inertial frame".

2.3 Tidal forces

The word "local" is essential in the above statements of the EP.
Consider a lab, freely falling radially towards the Earth, that contains two test particles at the same distance from the Earth but separated horizontally:

The gravitational attraction of the particles is tiny and can be neglected. Nevertheless, as the lab falls towards Earth, the particles will accelerate towards each other because the gravitational field has a slightly different direction at the location of the two particles. This is an example of a *tidal force*: a force arising from non-uniformity of the gravitational field. Such forces are physical: they cannot be eliminated by free fall.

2.4 Bending of light

The strong EP implies that light is bent by a gravitational field.
Consider a uniform gravitational field again. A freely falling laboratory is a local inertial frame. Inside the lab, the strong EP tells us that light rays must move on straight lines. But a straight line with respect to the lab corresponds to a curved path w.r.t to the original frame. In fact, this shows that light falls in the gravitational field in exactly the same way as a massive test particle: in time t is falls a distance $(1/2)gt^2$. (The effect is tiny: if the field is vertical then the time taken for the light to travel a horizontal distance d is $t = d/c$. In this time, the light falls a distance $h = gd^2/(2c^2)$. Taking $d = 1\,\mathrm{km}$, $g \approx 10\mathrm{ms}^{-2}$ gives $h \approx 5 \times 10^{-11}\mathrm{m}$.)

2.5 Gravitational red shift

Alice and Bob are at rest in a uniform gravitational field of strength g in the negative z-direction. Alice is at height $z = h$, Bob is at $z = 0$ (both are on the z-axis). They have identical clocks. Alice sends light signals to Bob at constant proper time intervals which she measures to be $\Delta\tau_A$. What is the proper time interval $\Delta\tau_B$ between the signals received by Bob?

Alice and Bob both have acceleration g with respect to a freely falling frame. Hence, by the EP, this experiment should give identical results to one in which Alice and Bob are moving with acceleration g in the positive z-direction in Minkowski spacetime. We choose our freely falling frame so that Alice and Bob are at rest at $t = 0$.

We shall neglect special relativistic effects in this problem, i.e., effects of order v^2/c^2 where v is a typical velocity (the analysis can extended to include such effects). The trajectories of Alice and Bob are therefore the usual Newtonian ones:

$$z_A(t) = h + \frac{1}{2}gt^2, \qquad z_B(t) = \frac{1}{2}gt^2 \tag{5}$$

Alice and Bob have $v = gt$ so we shall assume that gt/c is small over the time it takes to perform the experiment. We shall neglect effects of order g^2t^2/c^2.

Assume Alice emits the first light signal at $t = t_1$. Its trajectory is $z = z_A(t_1) - c(t - t_1) = h + (1/2)gt_1^2 - c(t - t_1)$ so it reaches Bob at time $t = T_1$ where this equals $z_B(T_1)$, i.e.,

$$h + \frac{1}{2}gt_1^2 - c(T_1 - t_1) = \frac{1}{2}gT_1^2 \tag{6}$$

The second light signal is emitted at time $t = t_1 + \Delta\tau_A$ (there is no special relativistic time dilation to the accuracy we are using here so the proper time interval $\Delta\tau_A$ is the same as an inertial time interval). Its trajectory is $z = z_A(t_1 + \Delta\tau_A) - c(t - t_1 - \Delta\tau_A)$. Let it reach Bob at time $t = T_1 + \Delta\tau_B$, i.e., the proper time intervals between the signals received by Bob is $\Delta\tau_B$. Then we have

$$h + \frac{1}{2}g(t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2}g(T_1 + \Delta\tau_B)^2. \tag{7}$$

Subtracting equation (6) gives

$$c(\Delta\tau_A - \Delta\tau_B) + \frac{g}{2}\Delta\tau_A(2t_1 + \Delta\tau_A) = \frac{g}{2}\Delta\tau_B(2T_1 + \Delta\tau_B) \tag{8}$$

The terms quadratic in $\Delta\tau_A$ and $\Delta\tau_B$ are negligible. This is because we must assume $g\Delta\tau_A \ll c$, since otherwise Alice would reach relativistic speeds by the time she emitted the second signal. Similarly for $\Delta\tau_B$.

We are now left with a linear equation relating $\Delta\tau_A$ and $\Delta\tau_B$

$$c(\Delta\tau_A - \Delta\tau_B) + g\Delta\tau_A t_1 = g\Delta\tau_B T_1 \tag{9}$$

Rearranging:

$$\Delta\tau_B = \left(1 + \frac{gT_1}{c}\right)^{-1}\left(1 + \frac{gt_1}{c}\right)\Delta\tau_A \approx \left(1 - \frac{g(T_1 - t_1)}{c}\right)\Delta\tau_A \tag{10}$$

where we have used the binomial expansion and neglected terms of order $g^2T_1^2/c^2$. Finally, to leading order we have $T_1 - t_1 = h/c$ (this is the time it takes the light to travel from A to B) and hence

$$\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A \tag{11}$$

The proper time between the signals received by Bob is *less* than that between the signals emitted by Alice. Time appears to run more slowly for Bob. For example, Bob will see that Alice ages more rapidly than him.
If Alice sends a pulse of light to Bob then we can apply the above argument to each successive wavecrest, i.e., $\Delta\tau_A$ is the period of the light waves. Hence $\Delta\tau_A = \lambda_A/c$ where λ_A is the wavelength of the light emitted by Alice. Bob receives light with wavelength λ_B where $\Delta\tau_B = \lambda_B/c$. Hence we have

$$\lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A. \tag{12}$$

The light received by Bob has shorter wavelength than the light emitted by Alice: it has undergone a *blueshift*. Light falling in a gravitational field is blueshifted.
This prediction of the EP was confirmed experimentally by the Pound-Rebka experiment (1960) in which light was emitted at the top of a tower and absorbed at the bottom. High accuracy was needed since $gh/c^2 = \mathcal{O}(10^{-15})$.
An identical argument reveals that light climbing out of a gravitational field undergoes a *redshift*. We can write the above formula in a form that applies to both situations:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A \tag{13}$$

where Φ is the gravitational potential.

2.6 Curved spacetime

The equivalence principle implies that if two test masses initially have the same position and velocity then they will follow exactly the same trajectory in a gravitational field, even if they have very different composition. (This is *not* true of other forces: in an electromagnetic field, bodies with different charge to mass ratio will follow different trajectories.) This suggested to Einstein that the trajectories of test masses in a gravitational field are determined

by the structure of spacetime alone and hence gravity should be described geometrically.
To see the idea, consider a spacetime in which the proper time between two infinitesimally nearby events is given not by the Minkowskian formula

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{14}$$

but instead by

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi(x,y,z)}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi(x,y,z)}{c^2}\right)(dx^2 + dy^2 + dz^2), \tag{15}$$

where $\Phi/c^2 \ll 1$. Let Alice have spatial position $\mathbf{x}_A = (x_A, y_A, z_A)$ and Bob have spatial position $\mathbf{x}_B$. Assume that Alice sends a light signal to Bob at time t_A and a second signal at time $t_A + \Delta t$. Let Bob receive the first signal at time t_B. What time does he receive the second signal? We haven't discussed how one determines the trajectory of the light ray but this doesn't matter. The above geometry does not depend on t. Hence the trajectory of the second signal must be the same as the first signal (whatever this is) but simply shifted by a time Δt:

Hence Bob receives the second signal at time $t_B + \Delta t$. The proper time interval between the signals sent by Alice is given by

$$\Delta\tau_A^2 = \left(1 + \frac{2\Phi_A}{c^2}\right)\Delta t^2, \tag{16}$$

where $\Phi_A \equiv \Phi(\mathbf{x}_A)$. (Note $\Delta x = \Delta y = \Delta z = 0$ because her signals are sent from the same spatial position.) Hence, using $\Phi/c^2 \ll 1$,

$$\Delta\tau_A = \left(1 + \frac{2\Phi_A}{c^2}\right)^{1/2} \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t. \tag{17}$$

Similarly, the proper time between the signals received by Bob is

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t. \tag{18}$$

Hence, eliminating Δt:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\left(1 + \frac{\Phi_A}{c^2}\right)^{-1}\Delta\tau_A \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A, \tag{19}$$

which is just equation (13). The difference in the rates of the two clocks has been explained by the geometry of spacetime. The geometry (15) is actually the geometry predicted by General Relativity outside a time-independent, non-rotating distribution of matter, at least when gravity is weak, i.e., $\Phi/c^2 \ll 1$. (This is true in the Solar System: $|\Phi|/c^2 = GM/(rc^2) \sim 10^{-5}$ at the surface of the Sun.)

3 Differentiable manifolds

3.1 Introduction

In Minkowski spacetime we usually use inertial frame coordinates (t, x, y, z) since these are adapted to the symmetries of the spacetime so using these coordinates simplifies the form of physical laws. However, a general spacetime has no symmetries and therefore no preferred set of coordinates. In fact, a single set of coordinates might not be sufficient to describe the spacetime. A simple example of this is provided by spherical polar coordinates (θ, ϕ) on the surface of the unit sphere S^2 in $\mathbb{R}^3$:

These coordinates are not well-defined at $\theta = 0, \pi$ (what is the value of ϕ there?). Furthermore, the coordinate ϕ is discontinuous at $\phi = 0$ or 2π.

To describe S^2 so that a pair of coordinates is assigned in a smooth way to every point, we need to use several overlapping sets of coordinates. Generalizing this example leads to the idea of a *manifold*. In GR, we assume that spacetime is a 4-dimensional differentiable manifold.

3.2 Definition of a manifold

You know how to do calculus on $\mathbb{R}^n$. How do you do calculus on a curved space, e.g., S^2? Locally, S^2 looks like $\mathbb{R}^2$ so one can carry over standard results. However, one has to confront the fact that it is impossible to use a single coordinate system on S^2. In order to do calculus we need our coordinates systems to "mesh together" in a smooth way. Mathematically, this is captured by the notion of a differentiable manifold:

Definition. An n-dimensional differentiable manifold is a set M together with a collection of subsets $\mathcal{O}_\alpha$ such that

1. $\bigcup_\alpha \mathcal{O}_\alpha = M$, i.e., the subsets $\mathcal{O}_\alpha$ cover M
2. For each α there is a one-to-one and onto map $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is an open subset of $\mathbb{R}^n$.
3. If $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ overlap, i.e., $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ then the map $\phi_\beta \circ \phi_\alpha^{-1}$ maps from $\phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\alpha \subset \mathbb{R}^n$ to $\phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\beta \subset \mathbb{R}^n$. We require that this map be smooth (infinitely differentiable).

Remarks.

1. The maps ϕ_α are called *charts* or *coordinate systems*. Sometimes we shall write $\phi_\alpha(p) = (x^1_\alpha(p), x^2_\alpha(p), \ldots x^n_\alpha(p))$ and refer to $x^i_\alpha(p)$ as the coordinates of p. The set of charts on M is called its *atlas*.

2. Strictly speaking, we have defined above the notion of a *smooth* manifold. If we replace "smooth" in the definition by C^k (k-times continuously differentiable) then we obtain a C^k-manifold. We shall always assume the manifold is smooth.

Examples.

1. $\mathbb{R}^n$: this is a manifold with atlas consisting of the single chart $\phi : (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n)$.

2. S^1: the unit circle, i.e., the subset of $\mathbb{R}^2$ given by $(\cos\theta, \sin\theta)$ with $\theta \in \mathbb{R}$. We can't define a chart by using $\theta \in [0, 2\pi)$ as a coordinate because $[0, 2\pi)$ is not open. Instead we define one chart by $\phi_1 : S^1 - (1,0) \to (0, 2\pi)$, $\phi_1(p) = \theta_1$ with θ_1 defined by:

   and we defined a second chart by $\phi_2 : S^1 - (-1,0) \to (-\pi, \pi)$, $\phi_2(p) = \theta_2$ where θ_2 is defined by:

Neither chart covers all of S^1 but together they form an atlas. The charts overlap on the "upper" semi-circle and on the "lower" semi-circle. On the first of these we have $\phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1$. On the second we have $\phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1 - 2\pi$. These are obviously smooth functions.

3. S^2: the two-dimensional sphere defined by the surface $x^2 + y^2 + z^2 = 1$ in Euclidean space. Introduce spherical polar coordinates in the usual way:

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta \tag{20}$$

these equations define $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ uniquely. Hence this defines a chart $\psi : \mathcal{O} \to \mathcal{U}$ where $\mathcal{O}$ is S^2 with the points $(0, 0, \pm 1)$ and the line of longitude $y = 0, x > 0$ removed:

and $\mathcal{U}$ is $(0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2$. We can define a second chart using a different set of spherical polar coordinates defined as follows:

$$x = -\sin\theta'\cos\phi', \qquad y = \cos\theta', \qquad z = \sin\theta'\sin\phi', \tag{21}$$

where $\theta' \in (0, \pi)$ and $\phi' \in (0, 2\pi)$ are uniquely defined by these equations. This is a chart $\psi : \mathcal{O}' \to \mathcal{U}'$, where $\mathcal{O}'$ is S^2 with the points $(\pm 1, 0, 0)$ and the line $z = 0, x < 0$ removed:

and $\mathcal{U}'$ is $(0, \pi) \times (0, 2\pi)$. Clearly $S^2 = \mathcal{O} \cup \mathcal{O}'$. The functions $\psi \circ \psi'^{-1}$ and $\psi' \circ \psi^{-1}$ are smooth on $\mathcal{O} \cap \mathcal{O}'$ so these two charts define an atlas for S^2.

A given set M may admit many atlases, e.g., one can simply add extra charts to an atlas. We don't want to regard this as producing a distinct manifold. We say that two atlases are *compatible* if their union is also an atlas. The union of all atlases compatible with a given atlas is called a *complete atlas*: it is an atlas which is not contained in any other atlas. We shall always assume that were are dealing with a complete atlas. (None of the above examples gives a complete atlas; such atlases necessarily contain infinitely many charts.)

4 Smooth functions

We shall need the notion of a smooth function on a smooth manifold. If $\phi : \mathcal{O} \to \mathcal{U}$ is a chart and $f : M \to \mathbb{R}$ then note that $f \circ \phi^{-1}$ is a map from $\mathcal{U}$, i.e., a subset of $\mathbb{R}^n$, to $\mathbb{R}$.

Definition. A function $f : M \to \mathbb{R}$ is *smooth* if, and only if, for any chart ϕ, $F \equiv f \circ \phi^{-1} : \mathcal{U} \to \mathbb{R}$ is a smooth function.

Remark. In GR, a function $f : M \to \mathbb{R}$ is sometimes called a *scalar field*.

Examples

1. Consider the example of S^1 discussed above. Let $f : S^1 \to \mathbb{R}$ be defined by $f(x, y) = x$ where (x, y) are the Cartesian coordinates in $\mathbb{R}^2$ labelling a point on S^1. In the first chart ϕ_1 we have $f \circ \phi_1^{-1}(\theta_1) = f(\cos\theta_1, \sin\theta_1) = \cos\theta_1$, which is smooth. Similary $f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2$ is also smooth. If ϕ is any other chart then we can write $f \circ \phi^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi^{-1})$, which is smooth because we've just seen that $f \circ \phi_i^{-1}$ are smooth, and $\phi_i \circ \phi^{-1}$ is smooth from the definition of a manifold. Hence f is a smooth function.

2. Consider a manifold M that is covered by a single chart ϕ but whose atlas also contains other charts ϕ_α. Let $\phi : p \mapsto (x^1(p), x^2(p), \ldots x^n(p))$. Then we can regard x^1 (say) as a function on M. Is it a smooth function? Yes: $x^1 \circ \phi_\alpha^{-1}$ is smooth for any chart ϕ_α, because it is the

first component of the map $\phi \circ \phi_\alpha^{-1}$, and the latter is smooth by the definition of a manifold.

3. Often it is convenient to define a function by specifying F instead of f. More precisely, given an atlas $\{\phi_\alpha\}$, we define f by specifying functions $F_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ and then setting $f = F_\alpha \circ \phi_\alpha$. One has to make sure that the resulting definition is independent of α on chart overlaps. For example, for S^1 using the atlas discussed above, define $F_1 : (0, 2\pi) \to \mathbb{R}$ by $\theta_1 \mapsto \sin(m\theta_1)$ and $F_2 : (-\pi, \pi) \to \mathbb{R}$ by $\theta_2 \mapsto \sin(m\theta_2)$, where m is an integer. On the chart overlaps we have $F_1 \circ \phi_1 = F_2 \circ \phi_2$ because θ_1 and θ_2 differ by a multiple of 2π on both overlaps. Hence this defines a function on S^1.

5 Curves and vectors

$\mathbb{R}^n$, or Minkowski spacetime, has the structure of a vector space, e.g., it makes sense to add the position vectors of points. One can view more general vectors, e.g., the 4-velocity of a particle, as vectors in the space itself. This structure does not extend to more general manifolds, e.g., S^2. So we need to discuss how to define vectors on manifolds.
For a surface in $\mathbb{R}^3$, the set of all vectors tangent to the surface at some point p defines the *tangent plane* to the surface at p:

This has the structure of a 2d vector space. Note that the tangent planes at two different points p and q are *different*. It does not make sense to compare a vector at p with a vector at q. For example: if one tried to define the sum of a vector at p and a vector at q then to which tangent plane would the sum belong?

On a surface, the tangent vector to a curve in the surface is automatically tangent to the surface. We take this as our starting point for defining vectors on a general manifold. We start by defining the notion of a curve in a manifold, and then the notion of a tangent vector to a curve at a point p. We then show that the set of all such tangent vectors at p forms a vector space $T_p(M)$. This is the analogue of the tangent plane to a surface but it makes no reference to any embedding into a higher-dimensional space.

Definition A smooth curve in a differentiable manifold M is a smooth function $\lambda : I \to M$, where I is an open interval in $\mathbb{R}$ (e.g. $(0,1)$ or $(-1,\infty)$). In other words, $\phi_\alpha \circ \lambda$ is a smooth map from I to $\mathbb{R}^n$ for all charts ϕ_α.

Let $f : M \to \mathbb{R}$ and $\lambda : I \to M$ be a smooth function and a smooth curve respectively. Then $f \circ \lambda$ is a map from I to $\mathbb{R}$. Hence we can take its derivative to obtain the rate of change of f along the curve:

$$\frac{d}{dt}\left[(f \circ \lambda)(t)\right] = \frac{d}{dt}\left[f(\lambda(t))\right] \tag{22}$$

In $\mathbb{R}^n$ we are used to the idea that the rate of change of f along the curve at a point p is given by the directional derivative $\mathbf{X}_p \cdot (\nabla f)_p$ where $\mathbf{X}_p$ is the tangent to the curve at p. Note that the vector $\mathbf{X}_p$ defines a linear map from the space of smooth functions on $\mathbb{R}^n$ to $\mathbb{R}$: $f \mapsto \mathbf{X}_p \cdot (\nabla f)_p$. This is how we define a tangent vector to a curve in a general manifold:

Definition. Let $\lambda : I \to M$ be a smooth curve with (wlog) $\lambda(0) = p$. The *tangent vector to* λ *at* p is the linear map X_p from the space of smooth functions on M to $\mathbb{R}$ defined by

$$X_p(f) = \left\{ \frac{d}{dt}\left[f(\lambda(t))\right] \right\}_{t=0} \tag{23}$$

Note that this satisfies two important properties: (i) it is linear, i.e., $X_p(f + g) = X_p(f) + X_p(g)$ and $X_p(\alpha f) = \alpha X_p(f)$ for any constant α; (ii) it satisfies the Leibniz rule $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$, where f and g are smooth functions and fg is their product.
If $\phi = (x^1, x^2, \ldots x^n)$ is a chart defined in a neighbourhood of p and $F \equiv f \circ \phi^{-1}$ then we have $f \circ \lambda = f \circ \phi^{-1} \circ \phi \circ \lambda = F \circ \phi \circ \lambda$ and hence

$$X_p(f) = \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \tag{24}$$

Note that (i) the first term on the RHS depends only on f and ϕ, and the second term on the RHS depends only on ϕ and λ; (ii) we are using the Einstein summation convention, i.e., μ is summed from 1 to n in the above expression.

Proposition. The set of all tangent vectors at p forms a n-dimensional vector space, the *tangent space* $T_p(M)$.

Proof. Consider curves λ and κ through p, wlog $\lambda(0) = \kappa(0) = p$. Let their tangent vectors at p be X_p and Y_p respectively. We need to define addition of tangent vectors and multiplication by a constant. let α and β be constants. We define $\alpha X_p + \beta Y_p$ to be the linear map $f \mapsto \alpha X_p(f) + \beta Y_p(f)$. Next we need to show that this linear map is indeed the tangent vector to a curve through p. Let $\phi = (x^1, \ldots, x^n)$ be a chart defined in a neighbourhood of p. Consider the following curve:

$$\nu(t) = \phi^{-1}\left[\alpha(\phi(\lambda(t)) - \phi(p)) + \beta(\phi(\kappa(t)) - \phi(p)) + \phi(p)\right] \tag{25}$$

Note that $\nu(0) = p$. Let Z_p denote the tangent vector to this curve at p. From equation (24) we have

$$\begin{aligned} Z_p(f) &= \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left\{\frac{d}{dt}\left[\alpha(x^\mu(\lambda(t)) - x^\mu(p)) + \beta(x^\mu(\kappa(t)) - x^\mu(p)) + x^\mu(p)\right]\right\}_{t=0} \\ &= \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left[\alpha\left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} + \beta\left(\frac{dx^\mu(\kappa(t))}{dt}\right)_{t=0}\right] \\ &= \alpha X_p(f) + \beta Y_p(f) \\ &= (\alpha X_p + \beta Y_p)(f). \end{aligned}$$

Since this is true for any smooth function f, we have $Z_p = \alpha X_p + \beta Y_p$ as required. Hence $\alpha X_p + \beta Y_p$ is tangent to the curve ν at p. It follows that the set of tangent vectors at p forms a vector space (the zero vector is realized by the curve $\lambda(t) = p$ for all t).

The next step is to show that this vector space is n-dimensional. To do this, we exhibit a basis. Let $1 \leq \mu \leq n$. Consider the curve λ_μ through p defined by

$$\lambda_\mu(t) = \phi^{-1}(x^1(p), \ldots, x^{\mu-1}(p), x^\mu(p) + t, x^{\mu+1}(p), \ldots, x^n(p)). \tag{26}$$

The tangent vector to this curve at p is denoted $\left(\frac{\partial}{\partial x^\mu}\right)_p$. To see why, note that, using equation (24)

$$\left(\frac{\partial}{\partial x^\mu}\right)_p (f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}. \tag{27}$$

The n tangent vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$ are linearly independent. To see why, assume that there exist constants α^μ such that $\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0$. Then, for any function f we must have

$$\alpha^\mu \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} = 0. \tag{28}$$

Choosing $f = x^\nu$, this reduces to $\alpha^\nu = 0$. Letting this run over all values of ν we see that all of the constants α^ν must vanish, which proves linear independence.
Finally we must prove that these tangent vectors span the vector space. This follows from equation (24), which can be rewritten

$$X_p(f) = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) \tag{29}$$

this is true for any f hence

$$X_p = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p, \tag{30}$$

i.e. X_p can be written as a linear combination of the n tangent vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$. These n vectors therefore form a basis for $T_p(M)$, which establishes that the tangent space is n-dimensional. QED.

Remark. The basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_p, \mu = 1, \dots n\}$ is chart-dependent: we had to choose a chart ϕ defined in a neighbourhood of p to define it. Choosing a different chart would give a different basis for $T_p(M)$. A basis defined this way is sometimes called a *coordinate basis*.

Definition. Let $\{e_\mu, \mu = 1 \dots n\}$ be a basis for $T_p(M)$ (not necessarily a coordinate basis). We can expand any vector $X \in T_p(M)$ as $X = X^\mu e_\mu$. We call the numbers X^μ the *components* of X with respect to this basis.

Example. Using the coordinate basis $e_\mu = (\partial/\partial x^\mu)_p$, equation (30) shows that the tangent vector X_p to a curve $\lambda(t)$ at p (where $t = 0$) has components

$$X_p^\mu = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0}. \tag{31}$$

Remark. Note the placement of indices. We shall sum over repeated indices if one such index appears "upstairs" (as a superscript, e.g., X^μ) and the other "downstairs" (as a subscript, e.g., e_μ). (The index μ on $\left(\frac{\partial}{\partial x^\mu}\right)_p$ is regarded as downstairs.) If an equation involves the same index more than twice, or twice but both times upstairs or both times downstairs (e.g. $X_\mu Y_\mu$) then a mistake has been made.

let's consider the relationship between different coordinate bases. Let $\phi = (x^1, \ldots, x^n)$ and $\phi' = (x'^1, \ldots, x'^n)$ be two charts defined in a neighbourhood of p. Then, for any smooth function f, we have

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}(f \circ \phi^{-1})\right)_{\phi(p)} \\ &= \left(\frac{\partial}{\partial x^\mu}[(f \circ \phi'^{-1}) \circ (\phi' \circ \phi^{-1})]\right)_{\phi(p)} \end{aligned}$$

Now let $F' = f \circ \phi'^{-1}$. This is a function of the coordinates x'. Note that the components of $\phi' \circ \phi^{-1}$ are simply the functions $x'^\mu(x)$, i.e., the primed coordinates expressed in terms of the unprimed coordinates. Hence what we have is easy to evaluate using the chain rule:

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}(F'(x'(x)))\right)_{\phi(p)} \\ &= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'(x')}{\partial x'^\nu}\right)_{\phi'(p)} \\ &= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p (f) \end{aligned}$$

Hence we have

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p \tag{32}$$

This expresses one set of basis vectors in terms of the other. Let X^μ and X'^μ denote the components of a vector with respect to the two bases. Then we have

$$X = X^\nu \left(\frac{\partial}{\partial x^\nu}\right)_p = X^\nu \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\mu}\right)_p \tag{33}$$

and hence

$$X'^\mu = X^\nu \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \tag{34}$$

Elementary treatments of GR usually *define* a vector to be a set of numbers $\{X^\mu\}$ that transforms according to this rule under a change of coordinates. More precisely, they usually call this a "contravariant vector".

6 Covectors

Recall the following from linear algebra:

Definition. Let V be a real vector space. The *dual space* V^* of V is the vector space of linear maps from V to $\mathbb{R}$.

Lemma. If V is n-dimensional then so is V^*. If $\{e_\mu,\, \mu = 1, \ldots, n\}$ is a basis for V then V^* has a basis $\{f^\mu,\, \mu = 1, \ldots, n\}$, the *dual basis* defined by $f^\mu(e_\nu) = \delta^\mu_\nu$ (if $X = X^\mu e_\mu$ then $f^\mu(X) = X^\nu f^\mu(e_\nu) = X^\mu$).

Since V and V^* have the same dimension, they are isomorphic. For example the linear map defined by $e_\mu \mapsto f^\mu$ is an isomorphism. But this is basis-dependent: a different choice of basis would give a different isomorphism. In contrast, there is a *natural* (basis-independent) isomorphism between V and $(V^*)^*$:

Theorem. If V is finite dimensional then $(V^*)^*$ is naturally isomorphic to V. The isomorphism is $\Phi : V \to (V^*)^*$ where $\Phi(X)(\omega) = \omega(X)$ for all $\omega \in V^*$.

Now we return to manifolds:

Definition. The dual space of $T_p(M)$ is denoted $T^*_p(M)$ and called the *cotangent space* at p. An element of this space is called a *covector (or 1-form) at* p. If $\{e_\mu\}$ is a basis for $T_p(M)$ and $\{f^\mu\}$ is the dual basis then we can expand a covector η as $\eta_\mu f^\mu$. η_μ are called the components of η.

Note that (i) $\eta(e_\mu) = \eta_\nu f^\nu(e_\mu) = \eta_\mu$; (ii) if $X \in T_p(M)$ then $\eta(X) = \eta(X^\mu e_\mu) = X^\mu \eta(e_\mu) = X^\mu \eta_\mu$ (note the placement of indices!)

Definition. Let $f : M \to \mathbb{R}$ be a smooth function. Define a covector $(df)_p$ by $(df)_p(X) = X(f)$ for any vector $X \in T_p(M)$. $(df)_p$ is the *gradient of f at p*.

Examples.

1. Let $(x^1, \ldots, x^n)$ be a coordinate chart defined in a neighbourhood of p, recall that x^μ is a smooth function (in this neighbourhood) so we can

take $f = x^\mu$ in the above definition to define n covectors $(dx^\mu)_p$. Note that

$$(dx^\mu)_p\left(\left(\frac{\partial}{\partial x^\nu}\right)_p\right) = \left(\frac{\partial x^\mu}{\partial x^\nu}\right)_p = \delta^\mu_\nu \tag{35}$$

Hence $\{(dx^\mu)_p\}$ is the dual basis of $\{(\partial/\partial x^\mu)_p\}$.

2. To explain why we call $(df)_p$ the gradient of f at p, observe that (using (27))

$$(df)_p = (df)_p\left(\left(\frac{\partial}{\partial x^\mu}\right)_p\right)(dx^\mu)_p = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}(dx^\mu)_p. \tag{36}$$

Hence, in a coordinate basis, $(df)_p$ has components $(\partial F/\partial x^\mu)_{\phi(p)}$.

Exercise. Consider two different charts $\phi = (x^1, \ldots, x^n)$ and $\phi' = (x'^1, \ldots, x'^n)$ defined in a neighbourhood of p. Show that

$$(dx^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi'(p)}(dx'^\nu)_p, \tag{37}$$

and hence that, if ω_μ and ω'_μ are the components of $\omega \in T^*_p(M)$ w.r.t. the two coordinate bases, then

$$\omega'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)}\omega_\nu. \tag{38}$$

Elementary treatements of GR take this as the *definition* of a covector, which they usually call a "covariant vector".

7 Abstract index notation

So far, we have used Greek letters $\mu, \nu, \ldots$ to denote components of vectors or covectors with respect to a basis. Equations involving such indices are assumed to hold *only in that basis*. For example an equation of the form $X^\mu = \delta^\mu_1$ says that, in a particular basis, a vector X has only a single non-vanishing component. This will not be true in other bases. Furthermore, if we were just presented with this equation, we would not even know whether

or not the quantities $\{X^\mu\}$ are the components of a vector or just a set of n numbers.
The abstract index notation uses Latin letters $a, b, c, \ldots$. A vector X is denoted X^a or X^b or X^c etc. The letter used in the superscript does not matter. What matters is that there *is* a superscript Latin letter. This tells us that the object in question is a vector. We emphasize: X^a represents the vector itself, *not* a component of the vector. Similarly we denote a covector η by η_a (or η_b etc).
If we have an equation involving abstract indices then we can obtain an equation valid in any particular basis simply by replacing the abstract indices by basis indices (e.g. $a \to \mu$, $b \to \nu$ etc.). For example, consider the quantity $\eta_a X^a$ in the abstract index notation. We see that this involves a covector η_a and a vector X^a. Furthermore, in any basis, this quantity is equal to $\eta_\mu X^\mu = \eta(X)$. Hence $\eta_a X^a$ is the abstract index way of writing $\eta(X)$. Similarly, if f is a smooth function then $X(f) = X^a (df)_a$.
Conversely, if one has an equation involving Greek indices but one knows that it is true for an *arbitrary* basis then one can replace the Greek indices with Latin letters.
Latin indices must respect the rules of the summation convention so equations of the form $\eta_a \eta_a = 1$ or $\eta_b = 2$ do not make sense.

8 Tensors

In Newtonian physics, you are familiar with the idea that certain physical quantities are described by tensors (e.g. the inertia tensor). You may have encountered the idea that the Maxwell field in special relativity is described by a tensor. Tensors are very important in GR because the curvature of spacetime is described with tensors. In this section we shall define tensors at a point p and explain some of their basic properties.

Definition. A tensor of type (r, s) at p is a multilinear map

$$T : T_p^*(M) \times \ldots \times T_p^*(M) \times T_p(M) \times \ldots \times T_p(M) \to \mathbb{R}. \tag{39}$$

where there are r factors of $T_p^*(M)$ and s factors of $T_p(M)$. (Multilinear means that the map is linear in each argument.)

In other words, given r covectors and s vectors, a tensor of type (r, s) produces a real number.

Examples.

1. A tensor of type $(0,1)$ is a linear map $T_p(M) \to \mathbb{R}$, i.e., it is a covector.

2. A tensor of type $(1,0)$ is a linear map $T_p^*(M) \to \mathbb{R}$, i.e., it is an element of $(T_p^*(M))^*$ but this is naturally isomorphic to $T_p(M)$ hence a tensor of type $(1,0)$ is a vector. To see how this works, given a vector $X \in T_p(M)$ we define a linear map $T_p^*(M) \to \mathbb{R}$ by $\eta \mapsto \eta(X)$ for any $\eta \in T_p^*(M)$.

3. We can define a $(1,1)$ tensor δ by $\delta(\omega, X) = \omega(X)$ for any covector ω and vector X.

Definition. Let T be a tensor of type (r,s) at p. If $\{e_\mu\}$ is a basis for $T_p(M)$ with dual basis $\{f^\mu\}$ then the *components* of T in this basis are the numbers

$$T^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} = T(f^{\mu_1}, f^{\mu_2}, \ldots, f^{\mu_r}, e_{\nu_1}, e_{\nu_2}, \ldots, e_{\nu_s}) \tag{40}$$

In the abstract index notation, we denote T by $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}$.

Remark. Tensors of type (r,s) at p can be added together and multiplied by a constant, hence they form a vector space. Since such a tensor has n^{r+s} components, it is clear that this vector space has dimension n^{r+s}.

Examples.

1. Consider the tensor δ defined above. Its components are

$$\delta^\mu{}_\nu = \delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \delta^\mu_\nu, \tag{41}$$

   where the RHS is a Kronecker delta. This is true in any basis, so in the abstract index notation we write δ as δ^a_b.

2. Consider a $(2,1)$ tensor. Let η and ω be covectors and X a vector. Then in our basis we have

$$T(\eta, \omega, X) = T(\eta_\mu f^\mu, \omega_\nu f^\nu, X^\rho e_\rho) = \eta_\mu\omega_\nu X^\rho T(f^\mu, f^\nu, e_\rho) = T^{\mu\nu}{}_\rho \eta_\mu\omega_\nu X^\rho \tag{42}$$

   Now the basis we chose was *arbitrary*, hence we can immediately convert this to a basis-independent equation using the abstract index notation:

$$T(\eta, \omega, X) = T^{ab}{}_c \eta_a \omega_b X^c. \tag{43}$$

   This formula generalizes in the obvious way to a (r,s) tensor.

We have discussed the transformation of vectors and covectors components under a change of *coordinate* basis. Let's now examine how tensor components transform under an *arbitrary* change of basis. Let $\{e_\mu\}$ and $\{e'_\mu\}$ be two bases for $T_p(M)$. Let $\{f^\mu\}$ and $\{f'^\mu\}$ denote the corresponding dual bases. Expanding the primed bases in terms of the unprimed bases gives

$$f'^\mu = A^\mu{}_\nu f^\nu, \qquad e'_\mu = B^\nu{}_\mu e_\nu \tag{44}$$

for some matrices $A^\mu{}_\nu$ and $B^\nu{}_\mu$. These matrices are related because:

$$\delta^\mu_\nu = f'^\mu(e'_\nu) = A^\mu{}_\rho f^\rho(B^\sigma{}_\nu e_\sigma) = A^\mu{}_\rho B^\sigma{}_\nu f^\rho(e_\sigma) = A^\mu{}_\rho B^\sigma{}_\nu \delta^\rho_\sigma = A^\mu{}_\rho B^\rho{}_\nu. \tag{45}$$

Hence $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$. For a change between coordinate bases, our previous results give

$$A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right), \qquad B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \tag{46}$$

and these matrices are indeed inverses of each other (from the chain rule).

Exercise. Show that under an arbitrary change of basis, the components of a vector X and a covector η transform as

$$X'^\mu = A^\mu{}_\nu X^\nu, \qquad \eta'_\mu = (A^{-1})^\nu{}_\mu \eta_\nu. \tag{47}$$

Show that the components of a $(2,1)$ tensor T transform as

$$T'^{\mu\nu}{}_\rho = A^\mu{}_\sigma A^\nu{}_\tau (A^{-1})^\lambda{}_\rho T^{\sigma\tau}{}_\lambda. \tag{48}$$

The corresponding result for a (r,s) tensor is an obvious generalization of this formula.

Given a (r,s) tensor, we can construct a $(r-1,s-1)$ tensor by *contraction*. This is easiest to demonstrate with an example.

Example. Let T be a tensor of type $(3,2)$. Define a new tensor S of type $(2,1)$ as follows

$$S(\omega,\eta,X) = T(f^\mu,\omega,\eta,e_\mu,X) \tag{49}$$

where $\{e_\mu\}$ is a basis and $\{f^\mu\}$ is the dual basis, ω and η are arbitrary covectors and X is an arbitrary vector. This definition is basis-independent because

$$\begin{aligned} T(f'^\mu,\omega,\eta,e'_\mu,X) &= T(A^\mu{}_\nu f^\nu,\omega,\eta,(A^{-1})^\rho{}_\mu e_\rho,X) \\ &= (A^{-1})^\rho{}_\mu A^\mu{}_\nu T(f^\nu,\omega,\eta,e_\rho,X) \\ &= T(f^\mu,\omega,\eta,e_\mu,X). \end{aligned}$$

The components of S and T are related by $S^{\mu\nu}{}_{\rho} = T^{\sigma\mu\nu}{}_{\sigma\rho}$ in any basis. Since this is true in any basis, we can write it using the abstract index notation as

$$S^{ab}{}_{c} = T^{dab}{}_{dc} \tag{50}$$

Note that there are other $(2,1)$ tensors that we can build from $T^{abc}{}_{de}$. For example, there is $T^{abd}{}_{cd}$, which corresponds to replacing the RHS of (49) with $T(\omega, \eta, f^{\mu}, X, e_{\mu})$. The abstract index notation makes it clear how many different tensors can be defined this way: we can define a new tensor by "contracting" any upstairs index with any downstairs index.

Another important way of constructing new tensors is by taking the product of two tensors:

Definition. If S is a tensor of type (p,q) and T is a tensor of type (r,s) then the *outer product* of S and T, denoted $S \otimes T$ is a tensor of type $(p+r, q+s)$ defined by

$$\begin{aligned}(S \otimes T)&(\omega_1, \ldots, \omega_p, \eta_1, \ldots, \eta_r, X_1, \ldots, X_q, Y_1, \ldots, Y_s) \\ &= S(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q) T(\eta_1, \ldots, \eta_r, Y_1, \ldots, Y_s)\end{aligned} \tag{51}$$

where $\omega_1, \ldots, \omega_p$ and $\eta_1, \ldots, \eta_r$ are arbitrary covectors and $X_1, \ldots, X_q$ and $Y_1, \ldots, Y_s$ are arbitrary vectors.

Exercise. Show that this definition is equivalent to

$$(S \otimes T)^{a_1 \ldots a_p b_1 \ldots b_r}{}_{c_1 \ldots c_q d_1 \ldots d_s} = S^{a_1 \ldots a_p}{}_{c_1 \ldots c_q} T^{b_1 \ldots b_r}{}_{d_1 \ldots d_s} \tag{52}$$

Exercise. Show that, in a coordinate basis, any $(2,1)$ tensor T at p can be written as

$$T = T^{\mu\nu}{}_{\rho} \left(\frac{\partial}{\partial x^{\mu}}\right)_p \otimes \left(\frac{\partial}{\partial x^{\nu}}\right)_p \otimes (dx^{\rho})_p \tag{53}$$

This generalizes in the obvious way to a (r,s) tensor.

Remark. You may be wondering why we write $T^{ab}{}_{c}$ instead of T^{ab}_{c}. At the moment there is no reason why we should not adopt the latter notation. However, it is convenient to generalize our definition of tensors slightly. We have defined a (r,s) tensor to be a linear map with $r+s$ arguments, where the first r arguments are covectors and the final s arguments are vectors. We can generalize this by allowing the covectors and vectors to appear in any order. For example, consider a $(1,1)$ tensor. This is a map $T_p^*(M) \times T_p(M) \to \mathbb{R}$.

But we could just as well have defined it to be a map $T_p(M) \times T_p^*(M) \to \mathbb{R}$. The abstract index notation allows us to distinguish these possibilities easily: the first would be written as $T^a{}_b$ and the second as $T_a{}^b$. $(2,1)$ tensors come in 3 different types: $T^{ab}{}_c$, $T^a{}_b{}^c$ and $T_a{}^{bc}$. Each type of of (r,s) tensor gives a vector space of dimension n^{r+s} but these vector spaces are naturally isomorphic so often one does not bother to distinguish between them.
There is a final type of tensor operation that we shall need: symmetrization and antisymmetrization. Consider a $(0,2)$ tensor T. We can define two other $(0,2)$ tensors S and A as follows:

$$S(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X)), \; A(X,Y) = \frac{1}{2}(T(X,Y) - T(Y,X)), \quad (54)$$

where X and Y are vectors at p. In abstract index notation:

$$S_{ab} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad A_{ab} = \frac{1}{2}(T_{ab} - T_{ba}). \quad (55)$$

In a basis, we can regard the components of T as a square matrix. The components of S and A are just the symmetric and antisymmetric parts of this matrix. It is convenient to introduce some notation to describe the operations we have just defined: we write

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}). \quad (56)$$

These operations can be applied to more general tensors. For example,

$$T^{(ab)c}{}_d = \frac{1}{2}(T^{abc}{}_d + T^{bac}{}_d). \quad (57)$$

We can also symmetrize or antisymmetrize on more than 2 indices. To symmetrize on n indices, we sum over all permutations of these indices and divide the result by $n!$ (the number of permutations). To antisymmetrize we do the same but we weight each term in the sum by the sign of the permutation. The indices that we symmetrize over must be either upstairs or downstairs, they cannot be a mixture. For example,

$$T^{(abc)d} = \frac{1}{3!}\left(T^{abcd} + T^{bcad} + T^{cabd} + T^{bacd} + T^{cbad} + T^{acbd}\right). \quad (58)$$

$$T^a{}_{[bcd]} = \frac{1}{3!}\left(T^a{}_{bcd} + T^a{}_{cdb} + T^a{}_{dbc} - T^a{}_{cbd} - T^a{}_{dcb} - T^a{}_{bdc}\right). \quad (59)$$

Sometimes we might wish to (anti)symmetrize over indices which are not adjacent. In this case, we use vertical bars to denote indices excluded from the (anti)symmetrization. For example,

$$T_{(a|bc|d)} = \frac{1}{2}\left(T_{abcd} + T_{dbca}\right). \tag{60}$$

Exercise. Show that $T^{(ab)}X_{[a|cd|b]} = 0$.

9 Tensor fields

So far, we have defined vectors, covectors and tensors at a single point p. However, in physics we shall need to consider how these objects vary in spacetime. This leads us to define vector, covector and tensor *fields*.

Definition. A vector field is a map X which maps any point $p \in M$ to a vector X_p at p. Given a vector field X and a function f we can define a new function $X(f) : M \to \mathbb{R}$ by $X(f) : p \mapsto X_p(f)$. The vector field X is *smooth* if this map is a smooth function for any smooth f.

Example. Given any coordinate chart $\phi = (x^1, \ldots, x^n)$, the vector field $\frac{\partial}{\partial x^\mu}$ is defined by $p \mapsto \left(\frac{\partial}{\partial x^\mu}\right)_p$. Hence

$$\left(\frac{\partial}{\partial x^\mu}\right)(f) : p \mapsto \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}, \tag{61}$$

where $F \equiv f \circ \phi^{-1}$. You should convince yourself that smoothness of f implies that the above map defines a smooth function. Therefore $\partial/\partial x^\mu$ is a smooth vector field. (Note that $(\partial/\partial x^\mu)$ usually won't be defined on the whole manifold M since the chart ϕ might not cover the whole manifold. So strictly speaking this is not a vector field on M but only on a subset of M. We shan't worry too much about this distinction.)

Remark. Since the vector fields $(\partial/\partial x^\mu)_p$ provide a basis for $T_p(M)$ at any point p, we can expand an arbitrary vector field as

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu}\right) \tag{62}$$

Since $\partial/\partial x^\mu$ is smooth, it follows that X is smooth if, and only if, its components X^μ are smooth functions.

Definition. A covector field is a map ω which maps any point $p \in M$ to a covector ω_p at p. Given a covector field and a vector field X we can define a function $\omega(X) : M \to \mathbb{R}$ by $\omega(X) : p \mapsto \omega_p(X_p)$. The covector field ω is *smooth* if this function is smooth for any smooth vector field X.

Example. Let f be a smooth function. We have defined $(df)_p$ above. Now we simply let p vary to define a covector field df. Let X be a smooth vector field and f a smooth function. Then $df(X) = X(f)$. This is a smooth function of p (because X is smooth). Hence df is a smooth covector field: the *gradient of* f.

Remark. Taking $f = x^\mu$ reveals that dx^μ is a smooth covector field.

Definition. A (r, s) tensor field is a map T which maps any point $p \in M$ to a (r, s) tensor T_p at p. Given r covector fields $\eta_1, \dots, \eta_r$ and s vector fields $X_1, \dots, X_s$ we can define a function $T(\eta_1, \dots, \eta_r, X_1, \dots, X_s) : M \to \mathbb{R}$ by $p \mapsto T_p((\eta_1)_p, \dots, (\eta_r)_p, (X_1)_p, \dots, (X_s)_p)$. The tensor field T is *smooth* if this function is smooth for any smooth covector fields $\eta_1, \dots, \eta_r$ and vector fields $X_1, \dots, X_r$.

Exercise. Show that a tensor field is smooth if, and only if, its components in a coordinate chart are smooth functions.

Remark. Henceforth we shall assume that all tensor fields that we encounter are smooth.

10 The commutator

Let X and Y be vector fields and f a smooth function. Since $Y(f)$ is a smooth function, we can act on it with X to form a new smooth function $X(Y(f))$. Does the map $f \mapsto X(Y(f))$ define a vector field? No, because $X(Y(fg)) = X(fY(g) + gY(f)) = fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f)$ so the Leibniz law is not satisfied. However, we can also define $Y(X(f))$ and the combination $X(Y(f)) - Y(X(f))$ *does* obey the Leibniz law (check!).

Definition. The *commutator* of two vector fields X and Y is the vector field $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{63}$$

for any smooth function f.

To see that this does indeed define a vector field, we can evaluate it in a coordinate chart:

$$\begin{aligned}
[X,Y](f) &= X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \\
&= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \\
&= X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu} \\
&= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial F}{\partial x^\mu} \\
&= [X,Y]^\mu \left(\frac{\partial}{\partial x^\mu}\right)(f)
\end{aligned}$$

where

$$[X,Y]^\mu = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right). \tag{64}$$

Since f is arbitrary, it follows that

$$[X,Y] = [X,Y]^\mu \left(\frac{\partial}{\partial x^\mu}\right). \tag{65}$$

The RHS is a vector field hence $[X,Y]$ is a vector field whose components in a coordinate basis are given by (64). (Note that we *cannot* write equation (64) in abstract index notation because it is valid only in a *coordinate* basis.)

Example. Let $X = \partial/\partial x^1$ and $Y = x^1 \partial/\partial x^2 + \partial/\partial x^3$. The components of X are constant so $[X,Y]^\mu = \partial Y^\mu/\partial x^1 = \delta^\mu_2$ so $[X,Y] = \partial/\partial x^2$.

Exercise. Show that (i) $[X,Y] = -[Y,X]$; (ii) $[X,Y+Z] = [X,Y]+[X,Z]$; (iii) $[X,fY] = f[X,Y] + X(f)Y$; (iv) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (the *Jacobi identity*). Here X, Y, Z are vector fields and f is a smooth function.

Remark. The components of $(\partial/\partial x^\mu)$ in the coordinate basis are either 1 or 0. It follows that

$$\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0. \tag{66}$$

Conversely, it can be shown that if $X_1, \ldots, X_m$ ($m \leq n$) are vector fields that are linearly independent at every point, and whose commutators all vanish, then, in a neighbourhood of any point p, one can introduce a coordinate chart $(x^1, \dots x^n)$ such that $X_i = \partial/\partial x^i$ $(i = 1, \dots, m)$ throughout this neighbourhood.

11 Integral curves

In fluid mechanics, the velocity of a fluid is described by a vector field $\mathbf{u}(\mathbf{x})$ in $\mathbb{R}^3$ (we are returning to Cartesian vector notation for a moment). Consider a particle suspended in the fluid with initial position $\mathbf{x}_0$. It moves with the fluid so its position $\mathbf{x}(\mathbf{t})$ satisfies

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0. \tag{67}$$

The solution of this differential equation is called the *integral curve* of the vector field $\mathbf{u}$ through $\mathbf{x}_0$. The definition extends straightforwardly to a vector field on a general manifold:

Definition. Let X be a vector field on M and $p \in M$. An *integral curve of* X *through* p is a curve through p whose tangent at every point is X.

Let λ denote an integral curve of X with (wlog) $\lambda(0) = p$. In a coordinate chart, this definition reduces to the initial value problem

$$\frac{dx^\mu(t)}{dt} = X^\mu(x(t)), \qquad x^\mu(0) = x^\mu_p. \tag{68}$$

(Here we are using the abbreviation $x^\mu(t) = x^\mu(\lambda(t))$.) Standard ODE theory guarantees that there exists a unique solution to this problem. Hence there is a unique integral curve of X through any point p.

Example. In a chart $\phi = (x^1, \dots, x^n)$, consider $X = \partial/\partial x^1 + x^1\partial/\partial x^2$ and take p to be the point with coordinates $(0, \dots, 0)$. Then $dx^1/dt = 1$, $dx^2/dt = x^1$. Solving the first equation and imposing the initial condition gives $x^1 = t$, then plugging into the second equation and solving gives $x^2 = t^2/2$. The other coords are trivial: $x^\mu = 0$ for $\mu > 2$, so the integral curve is $t \mapsto \phi^{-1}(t, t^2/2, \dots, 0)$.

12 Metrics

A metric captures the notion of *distance* on a manifold. We can motivate the required definition by considering the case of $\mathbb{R}^3$. Let $\mathbf{x}(t)$, $a < t < b$ be a curve in $\mathbb{R}^3$ (we're using Cartesian vector notation). Then the length of the curve is

$$\int_a^b dt \sqrt{\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}}. \tag{69}$$

Inside the integral we see the norm of the tangent vector $d\mathbf{x}/dt$, in other words the scalar product of this vector with itself. Therefore to define a notion of distance on a general manifold, we shall start by introducing a scalar product between vectors.

A scalar product maps a pair of vectors to a number. In other words, at a point p, it is a map $g : T_p(M) \times T_p(M) \to \mathbb{R}$. A scalar product should be linear in each argument. Hence g is a $(0, 2)$ tensor at p. We call g a *metric tensor*. There are a couple of other properties that g should also satisfy:

Definition. A metric tensor at $p \in M$ is a $(0, 2)$ tensor g with the following properties:

1. It is symmetric: $g(X, Y) = g(Y, X)$ for all $X, Y \in T_p(M)$ (i.e. $g_{ab} = g_{ba}$)
2. It is non-degenerate: $g(X, Y) = 0$ for all $Y \in T_p(M)$ if, and only if, $X = 0$.

Remark. Sometimes we shall denote $g(X, Y)$ by $\langle X, Y \rangle$ or $X \cdot Y$.

Since the components of g form a symmetric matrix, one can introduce a basis that diagonalizes g. Non-degeneracy implies that none of the diagonal elements is zero. By rescaling the basis vectors, one can arrange that the diagonal elements are all ± 1. In this case, the basis is said to be *orthonormal*. There are many such bases but a standard algebraic theorem (Sylvester's law of inertia) states that the number of positive and negative elements is independent of the choice of orthonormal basis. The number of positive and negative elements is called the *signature* of the metric.

In differential geometry, one is usually interested in *Riemannian* metrics, i.e., those with signature $+ + + \ldots +$, and hence g is positive definite. In GR, we are interested in *Lorentzian* metrics, i.e., those with signature $- + + \ldots +$. We want g to be defined over the whole manifold, so we assume it to be a tensor field.

Definition. A *Riemannian (Lorentzian) manifold* is a pair (M, g) where M is a differentiable manifold and g is a Riemannian (Lorentzian) metric tensor field. A Lorentzian manifold is sometimes called a *spacetime*.

Remark. On a Riemannian manifold, we can now define the length of a curve in exactly the same way as above: let $\lambda : (a,b) \to M$ be a smooth curve with tangent vector X. Then the length of the curve is

$$\int_a^b dt \sqrt{g(X,X)|_{\lambda(t)}} \tag{70}$$

Exercise. Given a curve $\lambda(t)$ we can define a new curve simply by changing the parametrization: let $t = t(u)$ with $dt/du > 0$ and $u \in (c,d)$ with $t(c) = a$ and $t(d) = b$. Show that: (i) the new curve $\kappa(u) \equiv \lambda(t(u))$ has tangent vector $Y^a = (dt/du)X^a$; (ii) the length of these two curves is the same, i.e., our definition of length is independent of parametrization.

In a coordinate basis, we have (cf equation (53))

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{71}$$

Often we use the notation ds^2 instead of g and abbreviate this to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{72}$$

This notation captures the intuitive idea of an infinitesimal distance ds being determined by infinitesimal coordinate separations dx^μ.

Examples.

1. In $\mathbb{R}^n = \{(x^1, \ldots, x^n)\}$, the *Euclidean metric* is

$$g = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \tag{73}$$

   $(\mathbb{R}^n, g)$ is callled *Euclidean space*. A coordinate chart which covers all of $\mathbb{R}^4$ and which the components of the metric are $\mathrm{diag}(1,1,\ldots,1)$ is called *Cartesian*.

2. In $\mathbb{R}^4 = \{(x^0, x^1, x^2, x^3)\}$, the *Minkowski metric* is

$$g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{74}$$

   $(\mathbb{R}^4, g)$ is called *Minkowski spacetime*. A coordinate chart which covers all of $\mathbb{R}^4$ and in which the components of the metric are $\eta_{\mu\nu} \equiv \mathrm{diag}(-1,1,1,1)$ everywhere is called an *inertial frame*.

3. On S^2, let (θ, ϕ) denote the spherical polar coordinate chart discussed earlier. The (unit) *round metric* on S^2 is

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{75}$$

i.e. in the chart (θ, ϕ), we have $g_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$. Note this is positive definite for $\theta \in (0, \pi)$, i.e., on all of this chart. However, this chart does not cover the whole manifold so the above equation does not determine g everywhere. We can give a precise definition by adding that, in the chart (θ', ϕ') discussed earlier, $g = d\theta'^2 + \sin^2\theta' d\phi'^2$. One can check that this does indeed define a smooth tensor field. (This metric is the one induced from the embedding of S^2 into 3d Euclidean space: it is the "pull-back" of the metric on Euclidean space - see later for the definition of pull-back.)

Definition. Since g_{ab} is non-degenerate, it must be invertible. The *inverse metric* is a symmetric $(2, 0)$ tensor field denoted g^{ab} and obeys

$$g^{ab} g_{bc} = \delta^a_c \tag{76}$$

Example. For the metric on S^2 defined above, in the chart (θ, ϕ) we have $g^{\mu\nu} = \mathrm{diag}(1, 1/\sin^2\theta)$.

Definition. A metric determines a natural isomorphism between vectors and covectors. Given a vector X^a we can define a covector $X_a = g_{ab} X^b$. Given a covector η_a we can define a vector $\eta^a = g^{ab}\eta_b$. These maps are clearly inverses of each other.

Remark. This isomorphism is the reason why covectors are not more familiar: we are used to working in Euclidean space using Cartesian coordinates, for which $g_{\mu\nu}$ and $g^{\mu\nu}$ are both the identity matrix, so the isomorphism appears trivial.

Definition. For a general tensor, abstract indices will be "lowered" by contracting with g_{ab} and "raised" by contracting with g^{ab}. Raising and lowering preserve the ordering of indices. The resulting tensor will be denoted by the same letter as the original tensor.

Example. Let T be a $(3, 2)$ tensor. Then $T^a{}_b{}^{cde} = g_{bf} g^{dh} g^{ej} T^{afc}{}_{hj}$.

13 Lorentzian signature

Remark. On a Lorentzian manifold, we take basis indices $\mu, \nu, \ldots$ to run from 0 to $n-1$.

At any point p of a Lorentzian manifold, we can choose an orthonormal basis $\{e_\mu\}$ so that $g(e_\mu, e_\nu) = \eta_{\mu\nu} \equiv \mathrm{diag}(-1, 1, \ldots, 1)$. Such a basis is far from unique. If $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$ is any other such basis then we have

$$\eta_{\mu\nu} = g(e'_\mu, e'_\nu) = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu g(e_\rho, e_\sigma) = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \eta_{\rho\sigma}. \qquad (77)$$

Hence

$$\eta_{\mu\nu} A^\mu{}_\rho A^\nu{}_\sigma = \eta_{\rho\sigma}. \qquad (78)$$

These are the defining equations of a *Lorentz transformation* in special relativity. Hence different orthonormal bases at p are related by Lorentz transformations. We saw earlier that the components of a vector at p transform as $X'^\mu = A^\mu{}_\nu X^\nu$. We are starting to recover the structure of special relativity *locally*, as required by the Equivalence Principle.

Definition. On a Lorentzian manifold (M, g), a non-zero vector $X \in T_p(M)$ is *timelike* if $g(X, X) < 0$, *null* (or *lightlike*) if $g(X, X) = 0$, and *spacelike* if $g(X, X) > 0$.

Remark. In an orthonormal basis at p, the metric has components $\eta_{\mu\nu}$ so the tangent space at p has exactly the same structure as Minkowski spacetime, i.e., null vectors at p define a light cone that separates timelike vectors at p from spacelike vectors at p:

Exercise. Let X^a, Y^b be non-zero vectors at p that are orthogonal, i.e., $g_{ab} X^a Y^b = 0$. Show that (i) if X^a is timelike then Y^a is spacelike; (ii) if X^a is null then Y^a is spacelike or null; (iii) if X^a is spacelike then Y^a can be spacelike, timelike, or null. (*Hint.* Choose an orthonormal basis to make the components of X^a as simple as possible.)

Definition. On a Riemannian manifold, the *norm* of a vector X is $|X| = \sqrt{g(X,X)}$ and the *angle* between two non-zero vectors X and Y (at the same point) is θ where $\cos\theta = g(X,Y)/(|X|\,|Y|)$. The same definitions apply to spacelike vectors on a Lorentzian manifold.

Definition. A curve in a Lorentzian manifold is said to be timelike if its tangent vector is everywhere timelike. Null and spacelike curves are defined similarly. (Most curves do not satisfy any of these definitions because e.g. the tangent vector can change from timelike to null to spacelike along a curve.)

Remark. The length of a spacelike curve can be defined in exactly the same way as on a Riemannian manifold (equation (70)). What about a timelike curve?

Definition. let $\lambda(u)$ be a timelike curve with $\lambda(0) = p$. Let X^a be the tangent to the curve. The *proper time* τ from p along the curve is defined by

$$\frac{d\tau}{du} = \sqrt{-\left(g_{ab}X^a X^b\right)_{\lambda(u)}}, \qquad \tau(0) = 0. \tag{79}$$

Remark. In a coordinate chart, $X^\mu = dx^\mu/du$ so this definition can be rewritten in the form

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu, \tag{80}$$

with the understanding that this is to be evaluated along the curve. Integrating the above equation along the curve gives the proper time from p to some other point $q = \lambda(u_q)$ as

$$\tau = \int_0^{u_q} du \sqrt{-\left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)_{\lambda(u)}} \tag{81}$$

Definition. If proper time τ is used to parametrize a timelike curve then the tangent to the curve is called the *4-velocity* of the curve. In a coordinate basis, it has components $u^\mu = dx^\mu/d\tau$.

Remark. (80) implies that 4-velocity is a unit timelike vector:

$$g_{ab}u^a u^b = -1. \tag{82}$$

14 Geodesics

Consider the following question. Let p and q be points connected by a timelike curve. A small deformation of a timelike curve remains timelike hence there exist infinitely many timelike curves connecting p and q. The proper time between p and q will be different for different curves. Which curve extremizes the proper time between p and q?

This is a standard Euler-Lagrange problem. Consider timelike curves from p to q with parameter u such that $\lambda(0) = p$, $\lambda(1) = q$. Let's use a dot to denote a derivative with respect to u. The proper time between p and q along such a curve is given by the functional

$$\tau[\lambda] = \int_0^1 du\, G\left(x(u), \dot{x}(u)\right) \tag{83}$$

where

$$G\left(x(u), \dot{x}(u)\right) \equiv \sqrt{-g_{\mu\nu}(x(u))\dot{x}^\mu(u)\dot{x}^\nu(u)} \tag{84}$$

and we are writing $x^\mu(u)$ as a shorthand for $x^\mu(\lambda(u))$.

Consider a 1-parameter family of timelike curves $\lambda_s(u)$ such that $\lambda_s(0) = p$, $\lambda_s(1) = q$ and λ_0 is the curve that extremizes the proper time. The proper time between p and q along λ_s is

$$\tau(s) \equiv \tau[\lambda_s] \tag{85}$$

Now, since $\lambda_0(u)$ extremizes the proper time, we have

$$0 = \left(\frac{d\tau}{ds}\right)_{s=0} = \int_0^1 du \left(\frac{\partial G}{\partial x^\mu}\frac{\partial x_s^\mu}{\partial s} + \frac{\partial G}{\partial \dot{x}^\mu}\frac{\partial^2 x^\mu}{\partial s \partial u}\right)_{s=0} \tag{86}$$

Interchange the s and u derivatives in the final term and then integrate by parts:

$$0 = \left[\frac{\partial G}{\partial \dot{x}^\mu}\frac{\partial x^\mu}{\partial s}\right]_0^1 + \int_0^1 \left(\frac{\partial G}{\partial x^\mu} - \frac{\partial}{\partial u}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right)\right)_{s=0}\left(\frac{\partial x_s^\mu}{\partial s}\right)_{s=0} \tag{87}$$

The first term vanishes because all of the curves have $x_s^\mu(0) = x^\mu(p)$ and so $\partial x^\mu/\partial s$ vanishes at $u = 0$ and similarly at $u = 1$. Now we want the curve with $s = 0$ to extremize the proper time for *any* 1-parameter family. Hence the integral above must vanish for any choice of $(\partial x_s^\mu/\partial s)_{s=0}$. It follows that the curve with $s = 0$, i.e., the curve that extremizes the proper time, must satisfy the Euler-Lagrange equation

$$\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0 \tag{88}$$

Working out the various terms, we have (using the symmetry of the metric)

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{2G} 2 g_{\mu\nu}\dot{x}^\nu = -\frac{1}{G} g_{\mu\nu}\dot{x}^\nu \tag{89}$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} g_{\nu\rho,\mu}\,\dot{x}^\nu \dot{x}^\rho \tag{90}$$

where we have relabelled some dummy indices, and introduced the important notation of a comma to denote partial differentiation:

$$g_{\nu\rho,\mu} \equiv \frac{\partial}{\partial x^\mu} g_{\nu\rho} \tag{91}$$

We will be using this notation a lot henceforth.
So far, our parameter u has been arbitrary subject to the conditions $u(0) = p$ and $u(1) = q$. At this stage, it is convenient to use a more physical parameter, namely τ, the proper time along the curve. (Note that we could not have used τ from the outset since the value of τ at q is different for different curves, which would make the range of integration different for different curves.) The paramers are related by

$$\left(\frac{d\tau}{du}\right)^2 = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = G^2 \tag{92}$$

and hence $d\tau/du = G$. So in our equations above, we can replace d/du with $Gd/d\tau$, so the Euler-Lagrange equation becomes (after cancelling a factor of $-G$)

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2} g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{93}$$

Hence

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} - \frac{1}{2} g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{94}$$

In the second term, we can replace $g_{\mu\nu,\rho}$ with $g_{\mu(\nu,\rho)}$ because it is contracted with an object symmetrical on ν and ρ. Finally, contracting the whole expression with the inverse metric and relabelling indices gives

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{95}$$

where $\Gamma^{\mu}_{\nu\rho}$ are known as the *Christoffel symbols*, and are defined by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right). \tag{96}$$

Remarks. $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$. The Christoffel symbols are *not* tensor components. Neither the first term nor the second term in (95) are components of a vector but the sum of these two terms does give vector components. More about this soon.

Example. In Minkowski spacetime, the components of the metric in an inertial frame are constant so $\Gamma^{\mu}_{\nu\rho} = 0$. Hence the above equation reduces to $d^2x^\mu/d\tau^2 = 0$. This is the equation of motion of a free particle! Hence, in Minkowski spacetime, the free particle trajectory between two (timelike separated) points p and q extremizes the proper time between p and q.

This motivates the following postulate of General Relativity:

Postulate. Massive free particles follow curves of extremal proper time, i.e., solutions of equation (95).

Definition. Solutions of equation (95) are called *geodesics*.

Remarks. 1. Massless particles obey a very similar equation which we shall discuss shortly. 2. In Minkowski spacetime, (timelike) geodesics *maximize* the proper time between two points. In a curved spacetime, this is true only *locally*.

Exercises

1. Show that the geodesic equation can be obtained more directly as the Euler-Lagrange equation for the Lagrangian

$$L = -g_{\mu\nu}(x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{97}$$

   This is usually the easiest way to derive the geodesic equation or to calculate the Christoffel symbols.

2. Note that L has no explicit τ dependence, i.e., $\partial L/\partial \tau = 0$. Show that this implies that the following quantity is conserved along geodesics (i.e. that is is annihilated by $d/d\tau$):

$$L - \frac{\partial L}{\partial (dx^\mu/d\tau)} \frac{dx^\mu}{d\tau} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{98}$$

This is a check on the consistency of the geodesic equation because the definition of τ as proper time implies that the RHS must be -1.

Example. Consider the metric we saw earlier with $c = 1$:

$$ds^2 = -\left(1 + 2\Phi(x, y, z)\right) dt^2 + \left(1 - 2\Phi(x, y, z)\right)\left(dx^2 + dy^2 + dz^2\right), \qquad (99)$$

where $\Phi \ll 1$. The Euler-Lagrange equation for t is

$$\frac{d}{d\tau}\left[(1 + 2\Phi)\frac{dt}{d\tau}\right] = 0. \qquad (100)$$

Expanding out the derivative and multiplying by $(1 + 2\Phi)^{-1} \approx (1 - 2\Phi)$, and neglecting terms quadratic in Φ gives (using the notation $x^1 = x, x^2 = y, x^3 = z$ and employing the summation convention for indices i, j, k)

$$\frac{d^2t}{d\tau^2} + 2\partial_i\Phi\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0, \qquad (101)$$

Hence we have

$$\Gamma^0_{0i} = \Gamma^0_{i0} = \partial_i\Phi, \qquad \Gamma^0_{00} = \Gamma^0_{ij} = 0. \qquad (102)$$

The Euler-Lagrange equation for x^i gives

$$\frac{d^2x^i}{d\tau^2} + \partial_i\Phi\left(\frac{dt}{d\tau}\right)^2 + \left(\partial_i\Phi\delta_{jk} - 2\partial_{(j}\Phi\delta_{k)i}\right)\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0, \qquad (103)$$

This gives

$$\Gamma^i_{00} = \partial_i\Phi, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = 0, \qquad \Gamma^i_{jk} = \partial_i\Phi\delta_{jk} - 2\partial_{(j}\Phi\delta_{k)i} \qquad (104)$$

Now consider a particle that is moving non-relativistically, i.e., with velocity small compared to the speed of light:

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \ll 1. \qquad (105)$$

From equation (80) it follows that

$$\frac{dt}{d\tau} \approx 1, \qquad (106)$$

so we can take $t \approx \tau + t_0$ (for some constant t_0) and the Euler-Lagrange equations for the spatial variables reduce to

$$\frac{d^2x^i}{dt^2} = -\partial_i \Phi. \tag{107}$$

This is the Newtonian equation of motion for a particle in a time-independent gravitational field. Hence the above spacetime geometry, together with the assumption that free particles move on geodesics, reproduces the predictions of Newtonian theory for weak, time-independent, gravitational fields and particles moving non-relativistically. Furthermore, we have automatically recovered the equality of gravitational and inertial mass, i.e., the weak equivalence principle is explained by the hypothesis that particles move on geodesics!

Remark. We would have obtained exactly the same result if we had replaced the term $1-2\Phi$ in the metric by $1-\alpha\Phi$ for any α. Agreement with Newtonian theory cannot distinguish these possibilities. $\alpha = 2$ is the value predicted by GR, and confirmed by precision tests (see later).

15 Covariant derivative

To formulate physical laws, we need to be able to differentiate tensor fields. For scalar fields, partial differentiation is fine: $f_{,\mu} \equiv \partial f/\partial x^\mu$ are the components of the covector field $(df)_a$. However, for tensor fields, partial differentiation is no good because the partial derivative of a tensor field does not give another tensor field:

Exercise. Let V^a be a vector field. In any coordinate chart, let $T^\mu{}_\nu = V^\mu{}_{,\nu} \equiv \partial V^\mu/\partial x^\nu$. Show that $T^\mu{}_\nu$ do *not* transform as tensor components under a change of chart.

The problem is that differentiation involves comparing a tensor at two infinitesimally nearby points of the manifold. But we have seen that this does not make sense: tensors at different points belong to different spaces. The mathematical structure that overcomes this difficulty is called a *covariant derivative* or *connection*.

Definition. A *covariant derivative* ∇ on a manifold M is a map sending every pair of smooth vector fields X, Y to a smooth vector field $\nabla_X Y$, with the following properties (where X, Y, Z are vector fields and f, g are functions)

$$\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z, \tag{108}$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \tag{109}$$

$$\nabla_X (fY) = f \nabla_X Y + (\nabla_X f) Y, \qquad \text{(Leibniz rule)}, \tag{110}$$

where the action of ∇ on functions is defined by

$$\nabla_X f = X(f). \tag{111}$$

Remark. (108) implies that, at any point, the map $\nabla Y : X \mapsto \nabla_X Y$ is a linear map from $T_p(M)$ to itself. Hence it defines a $(1,1)$ tensor (see examples sheet 1). More precisely, if $\eta \in T_p^*(M)$ and $X \in T_p(M)$ then we define $(\nabla Y)(\eta, X) \equiv \eta(\nabla_X Y)$.

Definition. let Y be a vector field. The *covariant derivative of* Y is the $(1,1)$ tensor field ∇Y. In abstract index notation we usually write $(\nabla Y)^a{}_b$ as $\nabla_b Y^a$ or $Y^a{}_{;b}$

Remarks.

1. Similarly we define $\nabla f : X \mapsto \nabla_X f = X(f)$. Hence $\nabla f = df$. We can write this as either $\nabla_a f$ or $f_{;a}$ or $\partial_a f$ or $f_{,a}$ (i.e. the covariant derivative reduces to the partial derivative when acting on a function).

2. Does the map $\nabla : X, Y \mapsto \nabla_X Y$ define a $(1,2)$ tensor field? No - equation (110) shows that this map is not linear in Y.

Example. Pick a coordinate chart on M. Let ∇ be the partial derivative in this chart. This satisfies all of the above conditions. This is not a very interesting example of a covariant derivative because it depends on choosing a particular chart: if we use a different chart then this covariant derivative will *not* be the partial derivative in the new chart.

Definition. In a basis $\{e_\mu\}$ the *connection components* $\Gamma^\mu_{\nu\rho}$ are defined by

$$\nabla_\rho e_\nu \equiv \nabla_{e_\rho} e_\nu = \Gamma^\mu_{\nu\rho} e_\mu \tag{112}$$

Example. The Christoffel symbols are the coordinate basis components of a certain connection, the *Levi-Civita connection*, which is defined on any manifold with a metric. More about this soon.

Write $X = X^\mu e_\mu$ and $Y = Y^\mu e_\mu$. Now

$$\begin{aligned}
\nabla_X Y &= \nabla_X (Y^\mu e_\mu) = X(Y^\mu) e_\mu + Y^\mu \nabla_X e_\mu \qquad \text{(Leibniz)} \\
&= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu \nabla_{X^\nu e_\nu} e_\mu \\
&= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu \qquad \text{by (108)} \\
&= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \Gamma^\rho_{\mu\nu} e_\rho \\
&= X^\nu \left(e_\nu (Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \right) e_\mu
\end{aligned} \tag{113}$$

and hence

$$(\nabla_X Y)^\mu = X^\nu e_\nu (Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho X^\nu \tag{114}$$

so

$$Y^\mu{}_{;\nu} = e_\nu (Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \tag{115}$$

In a coordinate basis, this reduces to

$$Y^\mu{}_{;\nu} = Y^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} Y^\rho \tag{116}$$

The connection components $\Gamma^\mu_{\nu\rho}$ are not tensor components:

Exercise (examples sheet 2). Consider a change of basis

$$e'_\mu = (A^{-1})^\nu{}_\mu e_\nu \tag{117}$$

Show that

$$\Gamma'^\mu_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma^\tau_{\lambda\sigma} + A^\mu{}_\tau (A^{-1})^\sigma{}_\rho e_\sigma ((A^{-1})^\tau{}_\nu) \tag{118}$$

The presence of the second term demonstrates that $\Gamma^\mu_{\nu\rho}$ are not tensor components. Hence neither term in the RHS of equation (116) transforms as a tensor. However, the *sum* of these two terms does transform as a tensor.

Exercise. Let ∇ and $\tilde{\nabla}$ be two different connections on M. Show that $\nabla - \tilde{\nabla}$ is a $(1,2)$ tensor field. You can do this either from the definition of a connection, or from the transformation law for the connection components.

The action of ∇ is extended to general tensor fields by the Leibniz property. If T is a tensor field of type (r,s) then ∇T is a tensor field of type $(r, s+1)$. For example, if η is a covector field then, for any vector fields X and Y, we define

$$(\nabla_X \eta)(Y) \equiv \nabla_X (\eta(Y)) - \eta(\nabla_X Y). \tag{119}$$

It is not obvious that this defines a $(0,2)$ tensor but we can see this as follows:

$$\begin{aligned}(\nabla_X\eta)(Y) &= \nabla_X(\eta_\mu Y^\mu) - \eta_\mu(\nabla_X Y)^\mu \\ &= X(\eta_\mu)Y^\mu + \eta_\mu X(Y^\mu) - \eta_\mu\left(X^\nu e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu}Y^\rho X^\nu\right)\end{aligned} \quad (120)$$

where we used (114). Now, the second and third terms cancel ($X = X^\nu e_\nu$) and hence (renaming dummy indices in the final term)

$$(\nabla_X\eta)(Y) = \left(X(\eta_\mu) - \Gamma^\rho_{\mu\nu}\eta_\rho X^\nu\right)Y^\mu, \quad (121)$$

which is linear in Y^μ so $\nabla_X\eta$ is a covector field with components

$$\begin{aligned}(\nabla_X\eta)_\mu &= X(\eta_\mu) - \Gamma^\rho_{\mu\nu}\eta_\rho X^\nu \\ &= X^\nu\left(e_\nu(\eta_\mu) - \Gamma^\rho_{\mu\nu}\eta_\rho\right)\end{aligned} \quad (122)$$

This is linear in X^ν and hence $\nabla\eta$ is a $(0,2)$ tensor field with components

$$\eta_{\mu;\nu} = e_\nu(\eta_\mu) - \Gamma^\rho_{\mu\nu}\eta_\rho \quad (123)$$

In a coordinate basis, this is

$$\eta_{\mu;\nu} = \eta_{\mu,\nu} - \Gamma^\rho_{\mu\nu}\eta_\rho \quad (124)$$

Now the Leibniz rule can be used to obtain the formula for the coordinate basis components of ∇T where T is a (r,s) tensor:

$$\begin{aligned}T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s;\rho} &= T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s,\rho} + \Gamma^{\mu_1}_{\sigma\rho}T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\sigma\rho}T^{\mu_1\ldots\mu_{r-1}\sigma}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^\sigma_{\nu_1\rho}T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \ldots - \Gamma^\sigma_{\nu_s\rho}T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\sigma}\end{aligned} \quad (125)$$

Exercise. Prove this result for a $(1,1)$ tensor.

Remark. We are using a comma and semi-colon to denote partial, and covariant, derivatives respectively. If more than one index appears after a comma or semi-colon then the derivative is to be taken with respect to all indices. The index nearest to comma/semi-colon is the *first* derivative to be taken. For example, $f_{,\mu\nu} = f_{,\mu,\nu} \equiv \partial_\nu\partial_\mu f$, and $X^a{}_{;bc} = \nabla_c\nabla_b X^a$ (we cannot use abstract indices for the first example since it is not a tensor). The second partial derivatives of a function commute: $f_{,\mu\nu} = f_{,\nu\mu}$ but for a covariant derivative this is not true in general. Set $\eta = df$ in (124) to get, in a coordinate basis,

$$f_{;\mu\nu} = f_{,\mu\nu} - \Gamma^\rho_{\mu\nu}f_{,\rho} \quad (126)$$

Antisymmetrizing gives

$$f_{;[\mu\nu]} = -\Gamma^{\rho}_{[\mu\nu]} f_{,\rho} \qquad \text{(coordinate basis)} \tag{127}$$

Definition. A connection ∇ is *torsion-free* if $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any function f. From (127), this is equivalent to

$$\Gamma^{\rho}_{[\mu\nu]} = 0 \qquad \text{(coordinate basis)} \tag{128}$$

Lemma. For a torsion-free connection, if X and Y are vector fields then

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{129}$$

Proof. Use a coordinate basis:

$$\begin{aligned} X^\nu Y^\mu{}_{;\nu} - Y^\nu X^\mu{}_{;\nu} &= X^\nu Y^\mu{}_{;\nu} + \Gamma^\mu_{\rho\nu} X^\nu Y^\rho - Y^\nu X^\mu{}_{;\nu} - \Gamma^\mu_{\rho\nu} Y^\nu X^\rho \\ &= [X, Y]^\mu + 2\Gamma^\mu_{[\rho\nu]} X^\nu Y^\rho \\ &= [X, Y]^\mu \end{aligned} \tag{130}$$

Hence the equation is true in a coordinate basis and therefore (as it is a tensor equation) it is true in any basis.

Remark. Even with zero torsion, the second covariant derivatives of a *tensor* field do *not* commute. More soon.

16 The Levi-Civita connection

On a manifold with a metric, the metric singles out a preferred connection:

Theorem. Let M be a manifold with a metric g. There exists a unique torsion-free connection ∇ such that the metric is covariantly constant: $\nabla g = 0$ (i.e. $g_{ab;c} = 0$). This is called the *Levi-Civita (or metric) connection.*

Proof. Let X, Y, Z be vector fields then

$$X(g(Y, Z)) = \nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{131}$$

where we used the Leibniz rule and $\nabla_X g = 0$ in the second equality. Permuting X, Y, Z leads to two similar identities:

$$Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \tag{132}$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \tag{133}$$

Add the first two of these equations and subtract the third to get (using the symmetry of the metric)

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= g(\nabla_X Y + \nabla_Y X, Z) \\ &- g(\nabla_Z X - \nabla_X Z, Y) \\ &+ g(\nabla_Y Z - \nabla_Z Y, X) \end{aligned} \tag{134}$$

The torsion-free condition implies

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{135}$$

Using this and the same identity with X, Y, Z permuted gives

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= 2g(\nabla_X Y, Z) - g([X,Y],Z) \\ &- g([Z,X],Y) + g([Y,Z],X) \end{aligned} \tag{136}$$

Hence

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}\left[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\right. \\ &+ \left. g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\right] \end{aligned} \tag{137}$$

This determines $\nabla_X Y$ uniquely because the metric is non-degenerate. It remains to check that it satisfies the properties of a connection. For example:

$$\begin{aligned} g(\nabla_{fX} Y, Z) &= \frac{1}{2}\left[fX(g(Y,Z)) + Y(fg(Z,X)) - Z(fg(X,Y))\right. \\ &+ \left. g([fX,Y],Z) + g([Z,fX],Y) - fg([Y,Z],X)\right] \\ &= \frac{1}{2}\left[fX(g(Y,Z)) + fY(g(Z,X)) + Y(f)g(Z,X)\right. \\ &- fZ(g(X,Y)) - Z(f)g(X,Y) + fg([X,Y],Z) - Y(f)g(X,Z) \\ &+ \left. fg([Z,X],Y) + Z(f)g(X,Y) - fg([Y,Z],X)\right] \\ &= \frac{f}{2}\left[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\right. \\ &+ \left. g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\right] \\ &= fg(\nabla_X Y, Z) = g(f\nabla_X Y, Z) \end{aligned} \tag{138}$$

and hence $g(\nabla_{fX}Y - f\nabla_X Y, Z) = 0$ for any vector field Z so, by the non-degeneracy of the metric, $\nabla_{fX}Y = f\nabla_X Y$.

Exercise. Show that $\nabla_X Y$ as defined by (137) satisfies the other properties required of a connection.

Remark. In differential geometry, this theorem is called the *fundamental theorem of Riemannian geometry* (although it applies for a metric of any signature).

Let's determine the components of the Levi-Civita connection in a coordinate basis (for which $[e_\mu, e_\nu] = 0$):

$$g(\nabla_\rho e_\nu, e_\sigma) = \frac{1}{2}\left[e_\rho(g_{\nu\sigma}) + e_\nu(g_{\sigma\rho}) - e_\sigma(g_{\rho\nu})\right], \tag{139}$$

that is

$$g(\Gamma^\tau_{\nu\rho} e_\tau, e_\sigma) = \frac{1}{2}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \tag{140}$$

The LHS is just $\Gamma^\tau_{\nu\rho} g_{\tau\sigma}$. Hence if we multiply the whole equation by the inverse metric $g^{\mu\sigma}$ we obtain

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \tag{141}$$

This is the same equation as we obtained earlier; we have now shown that the Christoffel symbols are the components of the Levi-Civita connection.

Remark. In GR, we take the connection to be the Levi-Civita connection. This is not as restrictive as it sounds: we saw above that the difference between two connections is a tensor field. Hence we can write any connection (even one with torsion) in terms of the Levi-Civita connection and a $(1, 2)$ tensor field. In GR we could regard the latter as a particular kind of "matter" field, rather than as part of the geometry of spacetime.

17 Geodesics (again)

Previously we defined timelike geodesics as curves that extremize the proper time between two points of a spacetime, and showed that this gives the equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x(\tau))\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{142}$$

where τ is the proper time along the curve. The tangent vector to the curve has components $X^\mu = dx^\mu/d\tau$. This is defined only along the curve. However, we can extend X^μ (in an arbitrary way) to a neighbourhood of the curve, so that X^μ becomes a vector field, and the geodesic is an integral curve of this vector field. The chain rule gives

$$\frac{d^2x^\mu}{d\tau^2} = \frac{dX^\mu(x(\tau))}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial X^\mu}{\partial x^\nu} = X^\nu X^\mu{}_{,\nu}. \tag{143}$$

Note that the LHS is independent of how we extend X^μ hence so must be the RHS. We can now write the geodesic equation as

$$X^\nu\left(X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho}X^\rho\right) = 0 \tag{144}$$

which is the same as

$$X^\nu X^\mu{}_{;\nu} = 0, \qquad \text{or} \qquad \nabla_X X = 0, \tag{145}$$

This is the way that geodesics are usually defined in differential geometry:

Definition. Let M be a manifold with a connection ∇. An *affinely parameterized geodesic* is an integral curve of a vector field X satisfying $\nabla_X X = 0$ (in a coordinate chart this is equivalent to (142)).

Remarks.

1. What do we mean by "affinely parameterized"? Consider a curve with parameter t whose tangent X satisfies the above definition. Let u be some other parameter for the curve, so $t = t(u)$ and $dt/du > 0$. Then the tangent vector becomes $Y = hX$ where $h = dt/du$. Hence

   $$\nabla_Y Y = \nabla_{hX}(hX) = h\nabla_X(hX) = h^2\nabla_X X + X(h)hX = fY, \tag{146}$$

   where $f = X(h) = dh/dt$. Hence $\nabla_Y Y = fY$ describes the same geodesic. In this case, the geodesic is *not* affinely parameterized.

   It always is possible to find an affine parameter so there is no loss of generality in restricting to affinely parameterized geodesics. Note that the new parameter also is affine iff $X(h) = 0$, i.e., h is constant. Then $u = at + b$ where a and b are constants with $a > 0$ ($a = h^{-1}$). Hence there is a 2-parameter family of affine parameters for any geodesic.

2. In GR, this definition (with ∇ the Levi-Civita connection) replaces our earlier definition in terms of curves that extremize proper time. The reason for this is that, although the two definitions are equivalent for timelike curves, the new definition applies for arbitrary (i.e. not necessarily timelike) curves. In a coordinate chart, the new definition reduces to (142), i.e., the same as for timelike geodesics. Hence the easiest way to determine geodesics (of the Levi-Civita connection) is to determine the geodesic equation using the Lagrangian (97).

Theorem. Let M be a manifold with a connection ∇. Let $p \in M$ and $X_p \in T_p(M)$. Then there exists a unique affinely parameterized geodesic through p with tangent vector X_p at p.
Proof. Choose a coordinate chart x^μ in a neighbourhood of p. Consider a curve parameterized by τ. It has tangent vector with components $X^\mu = dx^\mu/d\tau$. The geodesic equation is (142). We want the curve to satisfy the initial conditions

$$x^\mu(0) = x^\mu_p, \qquad \left(\frac{dx^\mu}{d\tau}\right)_{\tau=0} = X^\mu_p. \tag{147}$$

This is a coupled system of n ordinary differential equations for the n functions $x^\mu(t)$. Existence and uniqueness is guaranteed by the standard theory of ordinary differential equations.

Exercise. Let X be tangent to an affinely parameterized geodesic of the Levi-Civita connection. Show that $\nabla_X(g(X,X)) = 0$ and hence $g(X,X)$ is constant along the geodesic. Therefore the tangent vector cannot change e.g. from timelike to null along the geodesic.

Postulate. In GR, free particles move on geodesics (of the Levi-Civita connection). These are timelike for massive particles, and null for massless particles (e.g. photons).

Remark. In the timelike case we can use proper time as an affine parameter. This imposes the additional restriction $g(X,X) = -1$. If τ and τ' both are proper times along a geodesic then $\tau' = \tau + b$ (i.e. $a = 1$ above). In other words, clocks measuring proper time differ only by their choice of zero. In particular, they measure equal time intervals. Similarly in the spacelike case (or on a Riemannian manifold), we use arc length s as affine parameter, which gives $g(X,X) = 1$ and $s' = s + b$. In the null case, there is no analogue of proper time or arc length and so there is a 2-parameter ambiguity in affine parameterization.

18 Normal coordinates

Definition. Let M be a manifold with a connection ∇. Let $p \in M$. The *exponential map* from $T_p(M)$ to M is defined as the map which sends X_p to the point unit affine parameter distance along the geodesic through p with tangent X_p at p.

Remark. It can be shown that this map is one-to-one and onto *locally*, i.e., for X_p in a neighbourhood of the origin in $T_p(M)$.

Exercise. Let $0 \leq t \leq 1$. Show that the exponential map sends tX_p to the point affine parameter distance t along the geodesic through p with tangent X_p at p.

Definition. Let $\{e_\mu\}$ be a basis for $T_p(M)$. *Normal coordinates at* p are defined in a neighbourhood of p as follows. Pick q near p. Then the coordinates of q are X^μ where X^a is the element of $T_p(M)$ that maps to q under the exponential map.

Remark. From the above exercise, it follows that affinely parameterized geodesics through p are given in normal coordinates by $X^\mu(t) = tX_p^\mu$. Hence the geodesic equation reduces to

$$\Gamma^\mu_{\nu\rho}(X(t))X_p^\nu X_p^\rho = 0. \tag{148}$$

Evaluating at $t = 0$ gives that $\Gamma^\mu_{\nu\rho}(p)X_p^\nu X_p^\rho = 0$. But X_p is arbitrary, so it follows that

$$\Gamma^\mu_{(\nu\rho)}(p) = 0. \tag{149}$$

Hence, for a torsion-free connection, introducing normal coordinates at p has the effect of setting to zero the connection components at p. The connection components away from p *will not* vanish in general.
Now consider a manifold with a metric, and apply the above to the Levi-Civita connection. We then have, at p,

$$0 = 2g_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} = g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu} \tag{150}$$

Antisymmetrize on $\mu\nu$ and use the symmetry of the metric to obtain $g_{[\mu|\rho|,\nu]} = 0$, i.e, the final two terms above cancel, and hence

$$g_{\mu\nu,\rho} = 0 \qquad \text{at } p \tag{151}$$

Again, we emphasize, this is valid *only at the point p*. At any point, we can introduce normal coordinates to make the first partial derivatives of the metric vanish at that point.
Now consider $\partial/\partial X^1$ in normal coordinate at p. The integral curve through p of this vector field is $X^\mu(t) = (t, 0, 0, \ldots, 0)$ (since $X^\mu = 0$ at p). But, from the above, this is the same as the geodesic through p with tangent vector e_1 at p. It follows that $\partial/\partial X^1 = e_1$ at p (since both vectors are tangent to the curve at p). Similarly $\partial/\partial X^\mu = e_\mu$ at p. But the choice of basis $\{e_\mu\}$ was arbitrary. So we are free to choose $\{e_\mu\}$ to be an orthonormal basis. $\partial/\partial X^\mu$ then defines an orthonormal basis at p too. Hence, we can choose coordinates so that $g_{\mu\nu,\rho}(p) = 0$ and

$$g_{\mu\nu}(p) = \eta_{\mu\nu} \text{ (Lorentzian)} \qquad g_{\mu\nu}(p) = \delta_{\mu\nu} \text{ (Riemannian)} \tag{152}$$

In summary, on a Lorentzian (Riemannian) manifold, we can choose coordinates in the neighbourhood of any point p so that the components of the metric at p are the same as those of the Minkowski metric in inertial coordinates (Euclidean metric in Cartesian coordinates), and the first partial derivatives of the metric vanish at p.

Definition. In a Lorentzian manifold a *local inertial frame at p* is a set of normal coordinates at p with the above properties.

Thus the assumption that spacetime is a Lorentzian manifold leads to a precise mathematical definition of a local inertial frame.

19 Minimal coupling, equivalence principle

Now we can discuss the formulation of physical laws in curved spacetime. The key requirement is *general covariance*: the laws of physics should be independent of any choice of basis or coordinate chart.
In special relativity, we restrict attention to coordinate systems corresponding to inertial frames. The laws of physics should exhibit *special covariance*, i.e, take the same form in any inertial frame (this is the principle of relativity). There is a straightforward set of rules for converting such laws of physics into generally covariant laws:

1. Replace the Minkowski metric by a curved spacetime metric.

2. Replace partial derivatives with covariant derivatives (associated to the Levi-Civita connection). This rule is called *minimal coupling* in analogy with a similar rule for charged fields in electrodynamics.

3. Replace coordinate basis indices μ, ν etc (referring to an inertial frame) with abstract indices a, b etc.

Examples. Let x^μ denote the coordinates of an inertial frame, and $\eta^{\mu\nu}$ the inverse Minkowski metric (which has the same components as $\eta_{\mu\nu}$).

1. The simplest Lorentz invariant field equation is the wave equation for a scalar field Φ
$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0. \tag{153}$$
Follow the rules above to obtain the wave equation in a general spacetime:
$$g^{ab}\nabla_a\nabla_b\Phi = 0, \qquad \text{or} \qquad \nabla^a\nabla_a\Phi = 0 \qquad \text{or} \qquad \Phi_{;a}{}^{a} = 0. \tag{154}$$
A simple generalization of this equation is the *Klein-Gordon equation* describing a scalar field of mass m:
$$\nabla^a\nabla_a\Phi - m^2\Phi = 0. \tag{155}$$

2. In special relativity, the electric and magnetic fields are combined into an antisymmetric tensor $F_{\mu\nu}$. The electric and magnetic fields in an inertial frame are obtained by the rule (i, j, k take values from 1 to 3) $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk}B_k$. The (source-free) Maxwell equations take the covariant form
$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho} = 0, \qquad \partial_{[\mu}F_{\nu\rho]} = 0. \tag{156}$$
Hence in a curved spacetime, the electromagnetic field is described by an antisymmetric tensor F_{ab} satisfying
$$g^{ab}\nabla_a F_{bc} = 0, \qquad \nabla_{[a}F_{bc]} = 0. \tag{157}$$
The Lorentz force law for a particle of charge q and mass m in Minkowski spacetime is
$$\frac{d^2x^\mu}{d\tau^2} = \frac{q}{m}\eta^{\mu\nu}F_{\nu\rho}\frac{dx^\rho}{d\tau} \tag{158}$$

where τ is proper time. We saw previously that the LHS can be rewritten as $u^\nu \partial_\nu u^\mu$ where $u^\mu = dx^\mu/d\tau$ is the 4-velocity. Now following the rules above gives the generally covariant equation

$$u^b \nabla_b u^a = \frac{q}{m} g^{ab} F_{bc} u^c = \frac{q}{m} F^a{}_b u^b. \tag{159}$$

Note that this reduces to the geodesic equation when $q = 0$.

Remark. The rules above ensure that we obtain generally covariant equations. But how do we know they are the *right* equations? The strong equivalence principle states that, in a local inertial frame, the laws of physics should take the same form as in an inertial frame in Minkowski spacetime. But we saw above, that in a local inertial frame at p, $\Gamma^\mu_{\nu\rho}(p) = 0$ and hence (first) covariant derivatives reduce to partial derivatives at p. For example, $\nabla^\mu \nabla_\mu \Phi = g^{\mu\nu} \nabla_\mu \partial_\nu \Phi$ (in any chart) and, at p, this reduces to $\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi$ in a local inertial frame at p (since the metric at p is $\eta_{\mu\nu}$). Hence all of our generally covariant equations reduce to the equations of special relativity in a local inertial frame at any given point. The strong equivalence principle is satisfied automatically if we use the above rules. Nevertheless, there is still some scope for ambiguity, which arises from the possibility of including terms in an equation involving the curvature of spacetime (see later). These vanish identically in Minkowski spacetime. Sometimes, such terms are fixed by mathematical consistency. However, this is not always possible: there is no reason why it should be possible to derive laws of physics in curved spacetime from those in flat spacetime. The ultimate test is comparison with observations.

20 Energy-momentum tensor

in GR, the curvature of spacetime is related to the energy and momentum of matter. So we need to discuss how the latter concepts are defined in GR. We shall start by discussing the energy and momentum of particles.

In special relativity, associated to any particle is a scalar called its *rest mass* (or simply its mass) m. If the particle has 4-velocity u^μ (again x^μ denote inertial frame coordinates) then its 4-momentum is

$$P^\mu = m u^\mu \tag{160}$$

The time component of P^μ is the particle's energy and the spatial components are its 3-momentum with respect to the inertial frame.
If an observer at some point p has 4-velocity $v^\mu(p)$ then he measures the particle's energy, when the particle is at q, to be

$$E = -\eta_{\mu\nu} v^\mu(p) P^\nu(q). \tag{161}$$

The way to see this is to choose an inertial frame in which, at p, the observer is at rest at the origin, so $v^\mu(p) = (1,0,0,0)$ so E is just the time component of $P^\nu(q)$ in this inertial frame.
By the equivalence principle, GR should reduce to SR in a local inertial frame. Hence in GR we also associate a rest mass m to any particle and define the 4-momentum of a particle with 4-velocity u^a as

$$P^a = m u^a \tag{162}$$

Note that

$$g_{ab} P^a P^b = -m^2 \tag{163}$$

The EP implies that when the observer and particle *both* are at p then (161) should be valid so the observer measures the particle's energy to be

$$E = -g_{ab}(p) v^a(p) P^b(p) \tag{164}$$

However, an important difference between GR and SR is that there is no analogue of equation (161) for $p \neq q$. This is because $v^a(p)$ and $P^a(q)$ are vectors defined at different points, so they live in different tangent spaces. There is no way they can be combined to give a scalar quantity. An observer at p cannot measure the energy of a particle at q.
Now let's consider the energy and momentum of continuous distributions of matter.

Example. Consider Maxwell theory (without sources) in Minkowski spacetime. Pick an inertial frame and work in pre-relativity notation using Cartesian tensors. The electromagnetic field has energy density

$$\mathcal{E} = \frac{1}{8\pi}\left(E_i E_i + B_i B_i\right) \tag{165}$$

and the momentum density (or energy flux density) is given by the Poynting vector:

$$S_i = \frac{1}{4\pi} \epsilon_{ijk} E_j B_k. \tag{166}$$

The Maxwell equations imply that these satisfy the conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \partial_i S_i = 0. \tag{167}$$

The momentum flux density is described by the stress tensor:

$$t_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}\left(E_k E_k + B_k B_k\right)\delta_{ij} - E_i E_j - B_i B_j\right], \tag{168}$$

with the conservation law

$$\frac{\partial S_i}{\partial t} + \partial_j t_{ij} = 0. \tag{169}$$

If a surface element has area dA and normal n_i then the force exerted on this surface by the electromagnetic field is $t_{ij} n_j dA$.
In special relativity, these three objects are combined into a single tensor, called variously the "energy-momentum tensor", the "stress tensor", the "stress-energy-momentum tensor" etc. In an inertial frame it is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma}\,\eta_{\mu\nu}\right) \tag{170}$$

where we've raised indices with $\eta^{\mu\nu}$. Note that this is a symmetric tensor. It has components $T_{00} = \mathcal{E}$, $T_{0i} = -S_i$, $T_{ij} = t_{ij}$. The conservation laws above are equivalent to the single equation

$$\partial^\mu T_{\mu\nu} = 0. \tag{171}$$

The definition of the energy-momentum tensor extends naturally to GR:

Definition. The energy-momentum tensor of a Maxwell field in a general spacetime is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac} F_b{}^c - \frac{1}{4} F^{cd} F_{cd}\, g_{ab}\right) \tag{172}$$

Exercise (examples sheet 2). Show that Maxwell's equations imply that

$$\nabla^a T_{ab} = 0. \tag{173}$$

In GR (and SR) we assume that continuous matter always is described by a conserved energy-momentum tensor:

Postulate. The energy, momentum, and stresses, of matter are described by an *energy-momentum tensor*, a $(0,2)$ symmetric tensor T_{ab} that is *conserved*: $\nabla^a T_{ab} = 0$.

Remark. Let u^a be the 4-velocity of an observer $\mathcal{O}$ at p. Evaluating $T_{ab}(p)u^a u^b$ in a local inertial frame for which $u^\mu = (1,0,0,0)$ reveals that this quantity is the *energy density* of matter at p measured by $\mathcal{O}$. Similarly, the 4-vector $j^a = -T^a{}_b u^b$ is the *energy-momentum current* measured by $\mathcal{O}$. The component of j^a along u^a is the energy density, the components perpendicular to u^a are the momentum density. In more detail, if x^a is perpendicular to u^a (which implies that x^a is spacelike) then the component of momentum in the x^a direction measured by $\mathcal{O}$ is $j_a x^a$. The part of T_{ab} perpendicular to u^a describes the stress tensor of matter measured by $\mathcal{O}$. More precisely, if x^a and y^a both are perpendicular to u^a then the "xy component" of the stress tensor measured by $\mathcal{O}$ is $T_{ab}x^a y^b$.

Remark. In an inertial frame x^μ in Minkowski spacetime, local conservation of T_{ab} is equivalent to equations of the form (167) and (169). If one integrates these over a fixed volume V in surfaces of constant $t = x^0$ then one obtains global conservation equations. For example, integrating (167) over V gives

$$\frac{d}{dt}\int_V \mathcal{E} = -\int_S \mathbf{S}\cdot\mathbf{n}\,dA \tag{174}$$

where the surface S (with outward unit normal $\mathbf{n}$) bounds V. In words: the rate of increase of the energy of matter in V is equal to minus the energy flux across S. In a general curved spacetime, such an interpretation is not possible. This is because the gravitational field can do work on the matter in the spacetime. One might think that one could obtain global conservation laws in curved spacetime by introducing a definition of energy density etc for the gravitational field. This is a subtle issue. The gravitational field is described by the metric g_{ab}. By analogy with other fields, one might expect that the energy density of the gravitational field should be some expression quadratic in first derivatives of g_{ab}. But we have seen that we can choose normal coordinates to make the first partial derivatives of g_{ab} vanish at any given point. Gravitational energy certainly exists but not in a local sense. For example one can define the *total* energy (i.e. the energy of matter and the gravitational field) for certain spacetimes (this will be discussed in the black holes course).

Example. A *perfect fluid* is described by a 4-velocity vector field u^a, and two scalar fields ρ and p. The energy-momentum tensor is

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{175}$$

ρ and p are the energy density and pressure measured by an observer co-moving with the fluid, i.e., one with 4-velocity u^a (check: $T_{ab}u^a u^b = \rho + p - p = \rho$). The equations of motion of the fluid can be derived by conservation of T_{ab}:

Exercise (examples sheet 2). Show that, for a perfect fluid, $\nabla^a T_{ab} = 0$ is equivalent to

$$u^a \nabla_a \rho + (\rho + p)\nabla_a u^a = 0, \qquad (\rho + p)u^b \nabla_b u_a = -(g_{ab} + u_a u_b)\nabla^b p \tag{176}$$

These are relativistic generalizations of the mass conservation equation and Euler equation of non-relativistic fluid dynamics. Note that a pressureless fluid moves on timelike geodesics. This makes sense physically: zero pressure implies that the fluid particles are non-interacting and hence behave like free particles.

21 Parallel transport

On a general manifold there is no way of comparing tensors at different points. For example, we can't say whether a vector at p is the same as a vector at q. However, with a connection we can define a notion of "a tensor that doesn't change along a curve":

Definition. Let X^a be the tangent to a curve. A tensor field T is *parallelly transported along the curve* if $\nabla_X T = 0$.

Remarks.

1. Some times we say "parallelly propagated" instead of "parallelly transported".

2. A geodesic is a curve whose tangent vector is parallelly transported along the curve.

3. Let p be a point on a curve. If we specify T at p then the above equation determines T uniquely everywhere along the curve. For example,

consider a $(1,1)$ tensor. Introduce a chart in a neighbourhood of p. Let t be the parameter along the curve. The equation can be written

$$\frac{dT^\mu{}_\nu}{dt} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma = 0. \tag{177}$$

Standard ODE theory guarantees a unique solution given initial values for the components $T^\mu{}_\nu$.

If q is some other point on the curve then parallel transport along a curve from p to q determines an isomorphism between tensors at p and tensors at q.

If we use the Levi-Civita connection in Euclidean space or in Minkowski spacetime and we use Cartesian/inertial frame coordinates then the Christoffel symbols vanish everywhere. A tensor is parallelly transported along a curve iff its components are constant along the curve. Hence if we have two different curves from p to q then the result of parallelly transporting T from p to q is independent of which curve we choose. However, in a general spacetime this is no longer true: parallel transport is path-dependent. The path-dependence of parallel transport is measured by the *Riemann curvature tensor*. For Euclidean or Minkowski spacetime with the Levi-Civita connection, this vanishes and we say that the spacetime is *flat*.

22 The Riemann tensor

We shall return to the path-dependence of parallel transport below. First we define the Riemann tensor is as follows:

Definition. The *Riemann curvature tensor* $R^a{}_{bcd}$ of a connection ∇ is defined by $R^a{}_{bcd} Z^b X^c Y^d = (R(X,Y)Z)^a$, where X, Y, Z are vector fields and $R(X,Y)Z$ is the vector field

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{178}$$

To demonstrate that this defines a tensor, we need to show that it is linear in X, Y, Z. The symmetry $R(X,Y)Z = -R(Y,X)Z$ implies that we need only check linearity in X and Z. The non-trivial part is to check what happens if we multiply X or Z by a function f:

$$R(fX,Y)Z \;=\; \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$
\begin{aligned}
&= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - \nabla_{f[X,Y]}Z + \nabla_{Y(f)X}Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z \\
&= fR(X,Y)Z \qquad (179)
\end{aligned}
$$

$$
\begin{aligned}
R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\
&= \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z) \\
&\quad -f\nabla_{[X,Y]}Z - [X,Y](f)Z \\
&= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z \\
&\quad -f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z \\
&\quad -f\nabla_{[X,Y]}Z - [X,Y](f)Z \\
&= fR(X,Y)Z \qquad (180)
\end{aligned}
$$

It follows that our definition does indeed define a tensor. Let's calculate its components in a *coordinate* basis $\{e_\mu = \partial/\partial x^\mu\}$ (so $[e_\mu, e_\nu] = 0$). Use the notation $\nabla_\mu \equiv \nabla_{e_\mu}$,

$$
\begin{aligned}
R(e_\rho, e_\sigma)e_\nu &= \nabla_\rho\nabla_\sigma e_\nu - \nabla_\sigma\nabla_\rho e_\nu \\
&= \nabla_\rho(\Gamma^\tau_{\nu\sigma}e_\tau) - \nabla_\sigma(\Gamma^\tau_{\nu\rho}e_\tau) \\
&= \partial_\rho\Gamma^\mu_{\nu\sigma}e_\mu + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho}e_\mu - \partial_\sigma\Gamma^\mu_{\nu\rho}e_\mu - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}e_\mu \qquad (181)
\end{aligned}
$$

and hence, in a coordinate basis,

$$
R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} \qquad (182)
$$

Remark. It follows that the Riemann tensor vanishes for the Levi-Civita connection in Euclidean space or Minkowski spacetime (since one can choose coordinates for which the Christoffel symbols vanish everywhere).

The following contraction of the Riemann tensor plays an important role in GR:

Definition. The *Ricci curvature tensor* is the $(0,2)$ tensor defined by

$$
R_{ab} = R^c{}_{acb} \qquad (183)
$$

Exercise. Calculate the components of the Riemann and Ricci tensors for the static weak field metric (99), in the coordinate basis associated to (t, x^i), and working to first order in Φ. (Note that the Christoffel symbols given by equations (102) and (104) are $\mathcal{O}(\Phi)$ so you can neglect the $\Gamma\Gamma$ terms above.) Show that the non-zero components are:

$$R^0{}_{i0j} = -R^0{}_{ij0} = R^i{}_{00j} = -R^i{}_{0j0} = -\partial_i\partial_j\Phi \tag{184}$$

$$R^i{}_{jkl} = 2\delta_{i[k}\partial_{l]}\partial_j\Phi - 2\delta_{j[k}\partial_{l]}\partial_i\Phi \tag{185}$$

$$R_{00} = \partial_k\partial_k\Phi \qquad R_{ij} = \delta_{ij}\partial_k\partial_k\Phi \tag{186}$$

We saw earlier that, with vanishing torsion, the second covariant derivatives of a function commute. The same is not true of covariant derivatives of tensor fields. The failure to commute arises from the Riemann tensor:

Exercise. Let ∇ be a torsion-free connection. Prove the *Ricci identity*:

$$\nabla_c\nabla_d Z^a - \nabla_d\nabla_c Z^a = R^a{}_{bcd}Z^b \tag{187}$$

Hint. Show that the equation is true when multiplied by arbitrary vector fields X^c and Y^d.

23 Parallel transport again

Now we return to the relation between the Riemann tensor and the path-dependence of parallel transport. Let X and Y be vector fields that are linearly independent everywhere, with $[X,Y] = 0$. Earlier we saw that we can choose a coordinate chart $(s,t,\ldots)$ such that $X = \partial/\partial s$ and $Y = \partial/\partial t$. Let $p \in M$ and choose the coordinate chart such that p has coordinates $(0,\ldots,0)$. Let q,r,u be the point with coordinates $(\delta s, 0, 0, \ldots)$, $(\delta s, \delta t, 0, \ldots)$, $(0, \delta t, 0, \ldots)$ respectively, where δs and δt are small. We can connect p and q with a curve along which only s varies, with tangent X. Similarly, q and r can be connected by a curve with tangent Y. p and u can be connected by a curve with tangent Y, and u and r can be connected by a curve with tangent X. The result is a small quadrilateral:

Now let $Z_p \in T_p(M)$. Parallel transport Z_p along pqr to obtain a vector $Z_r \in T_r(M)$. Parallel transport Z_p along pur to obtain a vector $Z'_r \in T_r(M)$. We shall calculate the difference $Z'_r - Z_r$ for a torsion-free connection.

It is convenient (although not necessary) to introduce a new coordinate chart, namely normal coordinates at p. Henceforth, indices $\mu, \nu, \ldots$ will refer to this chart. s and t will now be used as *parameters* along the curves with tangent X and Y respectively.

pr is a curve with tangent vector X and parameter s. Along pr, Z is parallely propagated: $\nabla_X Z = 0$ so $dZ^\mu/ds = -\Gamma^\mu_{\nu\rho} Z^\nu X^\rho$ and hence $d^2Z^\mu/ds^2 = -(\Gamma^\mu_{\nu\rho} Z^\nu X^\rho)_{,\sigma} X^\sigma$. Now Taylor's theorem gives

$$\begin{aligned} Z^\mu_q &= Z^\mu_p + \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2 Z^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \\ &= Z^\mu_p - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu X^\rho X^\sigma\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \end{aligned} \tag{188}$$

where we have used $\Gamma^\mu_{\nu\rho}(p) = 0$ in normal coordinates at p. Now consider parallel transport along qr to obtain

$$\begin{aligned} Z^\mu_r &= Z^\mu_q + \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2 Z^\mu}{dt^2}\right)_p \delta t^2 + \mathcal{O}(\delta t^3) \\ &= Z^\mu_q - \left(\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho\right)_q \delta t - \frac{1}{2}\left((\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho)_{,\sigma} Y^\sigma\right)_q \delta t^2 + \mathcal{O}(\delta t^3) \\ &= Z^\mu_q - \left[\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho X^\sigma\right)_p \delta s + \mathcal{O}(\delta s^2)\right] \delta t \\ &\quad -\frac{1}{2}\left[\left((\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho Y^\sigma\right)_p + \mathcal{O}(\delta s)\right] \delta t^2 + \mathcal{O}(\delta t^3) \\ &= Z^\mu_p - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma}\right)_p \left[Z^\nu \left(X^\rho X^\sigma \delta s^2 + Y^\rho Y^\sigma \delta t^2 + 2 Y^\rho X^\sigma \delta s \delta t\right)\right]_p + \mathcal{O}(\delta^3) \end{aligned} \tag{189}$$

Here we assume that δs and δt both are $\mathcal{O}(\delta)$ (i.e. $\delta s = a\delta$ for some non-zero constant a and similarly for δt). Now consider parallel transport along pur. The result can be obtained from the above expression simply by interchanging X with Y and s with t. Hence we have

$$\begin{aligned} \Delta Z^\mu_r \equiv Z'^\mu_r - Z^\mu_r &= \left[\Gamma^\mu_{\nu\rho,\sigma} Z^\nu \left(Y^\rho X^\sigma - X^\rho Y^\sigma\right)\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\ &= \left[\left(\Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma}\right) Z^\nu X^\rho Y^\sigma\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\ &= \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right)_p \delta s \delta t + \mathcal{O}(\delta^3), \end{aligned} \tag{190}$$

where we used the expression (182) for the Riemann tensor components (remember that $\Gamma^\mu_{\nu\rho}(p) = 0$). We derived this in a coordinate basis defined using normal coordinates at p. But the LHS is a vector at r so we can transform to any other basis by multiplying by the matrix $A^\nu{}_\mu(r) = A^\nu{}_\mu(p) + \mathcal{O}(\delta)$. This has the effect of performing the corresponding basis transformation on the RHS. Hence our equation is basis-independent so we can write

$$\left(R^a{}_{bcd} Z^b X^c Y^d\right)_p = \lim_{\delta \to 0} \frac{\Delta Z^a_r}{\delta s \delta t} \tag{191}$$

The Riemann tensor does indeed measure the path-dependence of parallel transport.

Remark. We considered parallel transport along two different curves from p to r. However, we can reinterpret the result as describing the effect of parallel transport of a vector Z^a_r around the closed curve $rqpur$ to give the vector Z'^a_r. Hence ΔZ^a_r measures the change in Z^a_r when parallel transported around a closed curve.

24 Symmetries of the Riemann tensor

From its definition, we have the symmetry $R^a{}_{bcd} = -R^a{}_{bdc}$, equivalently:

$$R^a{}_{b(cd)} = 0. \tag{192}$$

Proposition. If ∇ is torsion-free then

$$R^a{}_{[bcd]} = 0. \tag{193}$$

Proof. Let $p \in M$ and choose normal coordinates at p. Vanishing torsion implies $\Gamma^\mu_{\nu\rho}(p) = 0$ and $\Gamma^\mu_{[\nu\rho]} = 0$ everywhere. We have $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$ at p. Antisymmetrizing on $\nu\rho\sigma$ now gives $R^\mu{}_{[\nu\rho\sigma]} = 0$ at p in the coordinate basis defined using normal coordinates at p. But if the components of a tensor vanish in one basis then they vanish in any basis. This proves the result at p. However, p is arbitrary so the result holds everywhere.

Proposition. (Bianchi identity). If ∇ is torsion-free then

$$R^a{}_{b[cd;e]} = 0 \tag{194}$$

Proof. Use normal coordinate at p again. At p,

$$R^{\mu}{}_{\nu\rho\sigma;\tau} = \partial_{\tau} R^{\mu}{}_{\nu\rho\sigma} \tag{195}$$

In normal coordinates at p, $\partial R = \partial\partial\Gamma - \Gamma\partial\Gamma$ and the latter terms vanish at p, we only need to worry about the former:

$$R^{\mu}{}_{\nu\rho\sigma;\tau} = \partial_{\tau}\partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\tau}\partial_{\sigma}\Gamma^{\mu}_{\nu\rho} \qquad \text{at } p \tag{196}$$

Antisymmetrizing gives $R^{\mu}{}_{\nu[\rho\sigma;\tau]} = 0$ at p in this basis. But again, if this is true in one basis then it is true in any basis. Furthermore, p is arbitrary. The result follows.

25 Geodesic deviation

Remark. In Euclidean space, or in Minkowski spacetime, initially parallel geodesics remain parallel forever. On a general manifold we have no notion of "parallel". However, we can study whether nearby geodesics move together or apart. In particular, we can quantify their "relative acceleration".

Definition. Let M be a manifold with a connection ∇. A *1-parameter family of geodesics* is a map $\gamma : I \times I' \to M$ where I and I' both are open intervals in $\mathbb{R}$, such that, for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t (so s is the parameter that labels the geodesic)
Let T be the tangent vector to the geodesics and also define S to be the vector tangent to the curves of constant t, which are parameterized by s:

In a chart x^{μ}, the geodesics are specified by $x^{\mu}(s,t)$ with $S^{\mu} = \partial x^{\mu}/\partial s$. Hence $x^{\mu}(s+\delta s, t) = x^{\mu}(s,t) + \delta s S^{\mu}(s,t) + \mathcal{O}(\delta s^2)$. Therefore $\delta s S^{a}$ is a *deviation vector* which points from one geodesic to an infinitesimally nearby one in the family.
The geodesics will fill out a 2-dimensional surface in our manifold. On this surface we can use s and t as coordinates. We can extend these to coordinates

$(s, t, \ldots)$ defined in a neighbourhood of the surface. This gives a coordinate chart in which $S = \partial/\partial s$ and $T = \partial/\partial t$ on the surface. We can now use these equations to define S and T throughout the neighbourhood of the surface, i.e., S and T are now vector fields, and satisfy the important property

$$[S, T] = 0 \tag{197}$$

Remark. If we fix attention on a particular geodesic then $V = \delta s \nabla_T S$ can be regarded as the rate of change of the relative position of a infinitesimally nearby geodesic in the family i.e., as the "relative velocity" of an infinitesimally nearby geodesic. We can define the "relative acceleration" of an infinitesimally nearby geodesic in the family as $A = \nabla_T V = \delta s \nabla_T \nabla_T S$. The word "relative" is important: the acceleration of a curve with tangent T is $\nabla_T T$, which vanishes here (as the curves are geodesics).

Proposition. If ∇ has vanishing torsion then

$$\nabla_T \nabla_T S = R(T, S)T \tag{198}$$

Proof. Vanishing torsion gives $\nabla_T S - \nabla_S T = [T, S] = 0$. Hence

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \nabla_T T + R(T, S)T, \tag{199}$$

where we used the definition of the Riemann tensor. But $\nabla_T T = 0$ because T is tangent to (affinely parameterized) geodesics.

Remark. This result is known as the *geodesic deviation equation*. In abstract index notation it is:

$$T^c \nabla_c (T^b \nabla_b S^a) = R^a{}_{bcd} T^b T^c S^d \tag{200}$$

This equation shows that curvature results in relative acceleration of geodesics. It also provides another method of measuring $R^a{}_{bcd}$: at any point p we can pick our 1-parameter family of geodesics such that T and S are arbitrary. Hence by measuring the LHS above we can determine $R^a{}_{(bc)d}$. From this we can determine $R^a{}_{bcd}$:

Exercise. Show that, for a torsion-free connection,

$$R^a{}_{bcd} = \frac{2}{3} \left(R^a{}_{(bc)d} - R^a{}_{(bd)c} \right) \tag{201}$$

Remarks.

1. Note that the relative acceleration vanishes for *all* families of geodesics if, and only if, $R^a{}_{bcd} = 0$.

2. In GR, free particles follow geodesics of the Levi-Civita connection. Geodesic deviation is the tendency of freely falling particles to move together or apart. We have already met this phenomenon: it arises from *tidal forces*. Hence the Riemann tensor is the quantity that measures tidal forces.

26 Curvature of the Levi-Civita connection

Remark. From now on, we shall restrict attention to a manifold with metric, and use the Levi-Civita connection. The Riemann tensor then enjoys additional symmetries. Note that we can use the metric to define R_{abcd}.

Proposition. The Riemann tensor satisfies

$$R_{abcd} = R_{cdab}, \qquad R_{(ab)cd} = 0. \tag{202}$$

Proof. The second identity follows from the first and the antisymmetry of the Riemann tensor. To prove the first, introduce normal coordinates at p, so $\partial_\mu g_{\nu\rho} = 0$ at p. Then, at p,

$$0 = \partial_\mu \delta^\nu_\rho = \partial_\mu \left(g^{\nu\sigma} g_{\sigma\rho} \right) = g_{\sigma\rho} \partial_\mu g^{\nu\sigma}. \tag{203}$$

Multiplying by the inverse metric gives $\partial_\mu g^{\nu\rho} = 0$ at p. Using this, we have

$$\partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2} g^{\tau\mu} \left(g_{\mu\nu,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho} \right) \qquad \text{at } p \tag{204}$$

And hence (as $\Gamma^\mu_{\nu\rho} = 0$ at p)

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} \right) \qquad \text{at } p \tag{205}$$

This satisfies $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ at p using the symmetry of the metric and the fact that partial derivatives commute. This establishes the identity in normal coordinates, but this is a tensor equation and hence valid in any basis. Furthermore p is arbitrary so the identity holds everywhere.

Proposition. The Ricci tensor is symmetric:

$$R_{ab} = R_{ba} \tag{206}$$

Proof. $R_{ab} = g^{cd} R_{dacb} = g^{cd} R_{cbda} = R^d{}_{bca} = R_{ba}$ where we used the first identity above in the second equality.

Definition. The *Ricci scalar* is

$$R = g^{ab} R_{ab} \tag{207}$$

Definition. The *Einstein tensor* is the symmetric $(0, 2)$ tensor defined by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{208}$$

Proposition. The Einstein tensor satisfies the *contracted Bianchi identity*:

$$\nabla^a G_{ab} = 0 \tag{209}$$

which can also be written as

$$\nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0 \tag{210}$$

Proof. Examples sheet 2.

27 Einstein's equation

Postulates of General Relativity.

1. Spacetime is a 4d Lorentzian manifold equipped with the Levi-Civita connection.
2. Free particles follow timelike or null geodesics.
3. The energy, momentum, and stresses of matter are described by a symmetric tensor T_{ab} that is conserved: $\nabla^a T_{ab} = 0$.

4. The curvature of spacetime is related to the energy-momentum tensor of matter by the *Einstein equation* (1915)

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G\, T_{ab} \tag{211}$$

where G is Newton's constant.

We have discussed points 1-3 above. It remains to discuss the Einstein equation. We can motivate this as follows. In GR, the gravitational field is described by the curvature of spacetime. Since the energy of matter should be responsible for gravitation, we expect *some* relationship between curvature and the energy-momentum tensor. The simplest possibility is a linear relationship, i.e., a curvature tensor is proportional to T_{ab}. Since T_{ab} is symmetric, it is natural to expect the Ricci tensor to be the relevant curvature tensor.

Einstein's first guess for the field equation of GR was $R_{ab} = CT_{ab}$ for some constant C. This does not work for the following reason. The RHS is conserved hence this equation implies $\nabla^a R_{ab} = 0$. But then from the contracted Bianchi identity we get $\nabla_a R = 0$. Taking the trace of the equation gives $R = CT$ (where $T = T^a{}_a$) and hence we must have $\nabla_a T = 0$, i.e., T is constant. But, T vanishes in empty space and is usually non-zero inside matter. Hence this is unsatisfactory.

The solution to this problem is obvious once one knows of the contracted Bianchi identity. Take G_{ab}, rather than R_{ab}, to be proportional to T_{ab}. The coefficient of proportionality on the RHS of Einstein's equation is fixed by demanding that the equation reduces to Newton's law of gravitation when the gravitational field is weak and the matter is moving non-relativistically. Let's now check that this is indeed the case.

Consider the static weak field metric

$$ds^2 = -(1 + 2\Phi(x,y,z))dt^2 + (1 - 2\Phi(x,y,z))(dx^2 + dy^2 + dz^2) \qquad |\Phi| \ll 1 \tag{212}$$

We have seen previously that the geodesic equation in this background reduces to Newton's laws of motion for a particle in a gravitational field, assuming non-relativistic motion. Let's assume that the gravitational field is due to a perfect fluid. Since the gravitational field is time-independent, it is natural to assume that the fluid producing it also is time-independent, i.e., the fluid is at rest: $u^\mu = (u^0, 0, 0, 0)$ in the above coordinates. Since

$g_{\mu\nu}u^\mu u^\nu = -1$ we have $u^0 = 1 + \mathcal{O}(\Phi)$. Similarly $u_0 = -1 + \mathcal{O}(\Phi)$. Hence, to lowest order in Φ,

$$T_{00} = \rho \qquad T_{0i} = 0 \qquad T_{ij} = p\,\delta_{ij} \tag{213}$$

The Ricci tensor of this metric was given above:

$$R_{00} = \partial_k \partial_k \Phi \qquad R_{0i} = 0 \qquad R_{ij} = \delta_{ij}\partial_k \partial_k \Phi \tag{214}$$

The Ricci scalar is $R = 2\partial_k \partial_k \Phi$ to leading order and hence the Einstein tensor is, to leading order,

$$G_{00} = 2\partial_k \partial_k \Phi \qquad G_{0i} = 0 \qquad G_{ij} = 0 \tag{215}$$

Hence the 00 component of Einstein's equation is

$$\partial_k \partial_k \Phi = 4\pi G\,\rho \tag{216}$$

which is Newton's law of gravity! The $0i$ component of Einstein's equation is trivial. However, the ij component gives $p = 0$. This looks bad but remember we are working only to leading order, so this equation just implies that the pressure must be subleading compared to the energy density ρ. This is indeed the case under all but the most extreme circumstances: reinserting factors of c then p/c^2 has the same dimensions as ρ. Under normal circumstances, $p/c^2 \ll \rho$ and hence the gravitational effect of fluid pressure is negligible compared to the gravitational effect of fluid energy density.

Remarks.

1. We've shown that the static weak field metric is *a* solution of Einstein's equation to leading order. Later we will show that it is the *unique* solution describing a weak, time-independent, gravitational field.

2. In vacuum, $T_{ab} = 0$ so Einstein's equation gives $G_{ab} = 0$. Contracting indices gives $R = 0$. Hence the *vacuum Einstein equation* can be written as
$$R_{ab} = 0 \tag{217}$$

3. The "geodesic postulate" of GR is redundant. Using conservation of the energy-momentum tensor it can be shown that a distribution of matter that is small (compared to the scale on which the spacetime metric varies), and sufficiently weak (so that its gravitational effect is small), must follow a geodesic.

4. The Einstein equation is a set of non-linear, second order, coupled, partial differential equations for the components of the metric $g_{\mu\nu}$. Very few physically interesting explicit solutions are known so one has to develop other methods to solve the equation, e.g., numerical integration.

5. How unique is the Einstein equation? Is there any tensor, other than G_{ab} that we could have put on the LHS? The answer is supplied by:

Theorem (Lovelock 1972). In a 4d spacetime, let H_{ab} be a symmetric tensor such that (i) in any coordinate chart, at any point, $H_{\mu\nu}$ is a function of $g_{\mu\nu}$, $g_{\mu\nu,\rho}$ and $g_{\mu\nu,\rho\sigma}$ at that point; (ii) $\nabla^a H_{ab} = 0$. Then there exist constants α and β such that

$$H_{ab} = \alpha G_{ab} + \beta g_{ab} \tag{218}$$

Hence (as Einstein realized) there is the freedom to add a constant multiple of g_{ab} to the LHS of Einstein's equation, giving

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{219}$$

Λ is called the *cosmological constant*. This no longer reduces to Newtonian theory for slow motion in a weak field but the deviation from Newtonian theory is unobservable if Λ is sufficiently small. Note that $|\Lambda|^{-1/2}$ has the dimensions of length. The effects of Λ are negligible on lengths or times small compared to this quantity. Astronomical observations suggest that there is indeed a very small positive cosmological constant: $\Lambda^{-1/2} \sim 10^9$ light years, the same order of magnitude as the size of the observable Universe. Hence the effects of the cosmological constant are negligible except on cosmological length scales. Therefore we shall set $\Lambda = 0$ until we discuss cosmology.
Note that we can move the cosmological constant term to the RHS of the Einstein equation, and regard it as the energy-momentum tensor of a perfect fluid with $\rho = -p = \Lambda/(8\pi G)$. Hence the cosmological constant is sometimes referred to as *dark energy* or *vacuum energy*. It is a great mystery why it is so small because arguments from quantum field theory suggest that it should be 10^{120} times larger. This is the *cosmological constant problem*. One (controversial) proposed solution of this problem invokes the *anthropic principle*, which posits the existence of many possible universes with different values for constants such as Λ. If Λ was very different from its observed value then galaxies never would have formed and hence we would not be here.

Remark. We have explicitly written Newton's constant G throughout this section. Henceforth we shall choose units so that $G = c = 1$.

28 Maps between manifolds

Definition. Let M, N be differentiable manifolds of dimension m, n respectively. A function $\phi : M \to N$ is *smooth* if, and only if, $\psi_A \circ \phi \circ \psi_\alpha^{-1}$ is smooth for all charts ψ_α of M and all charts ψ_A of N (note that this is a map from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$).

If we have such a map then we can "pull-back" a function on N to define a function on M:

Definition. Let $\phi : M \to N$ and $f : N \to \mathbb{R}$ be smooth functions. The *pull-back* of f by ϕ is the function $\phi^*(f) : M \to \mathbb{R}$ defined by $\phi^*(f) = f \circ \phi$, i.e., $\phi^*(f)(p) = f(\phi(p))$.

Furthermore, ϕ allows us to "push-forward" a curve λ in M to a curve $\phi \circ \lambda$ in N. Hence we can push-forward vectors from M to N:

Definition. Let $\phi : M \to N$ be smooth. Let $p \in M$ and $X \in T_p(M)$. The *push-forward* of X with respect to ϕ is the vector $\phi_*(X) \in T_{\phi(p)}(N)$ defined as follows. Let λ be a smooth curve in M passing through p with tangent X at p. Then $\phi_*(X)$ is the tangent vector to the curve $\phi \circ \lambda$ in N at the point $\phi(p)$.

Lemma. Let $f : N \to \mathbb{R}$. Then $(\phi_*(X))(f) = X(\phi^*(f))$.
Proof. Wlog $\lambda(0) = p$.

$$\begin{aligned}(\phi_*(X))(f) &= \left[\frac{d}{dt}(f \circ (\phi \circ \lambda))(t)\right]_{t=0} \\ &= \left[\frac{d}{dt}(f \circ \phi) \circ \lambda)(t)\right]_{t=0} \\ &= X(\phi^*(f)) \end{aligned} \tag{220}$$

Exercise. Let x^μ be coordinates on M and y^α be coordinates on N (we use different indices α, β etc for N because N is a different manifold which might not have the same dimension as M). Then we can regard ϕ as defining a map $y^\alpha(x^\mu)$. Show that the components of $\phi^*(X)$ are related to those of X by

$$(\phi_*(X))^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p X^\mu \tag{221}$$

The map on covectors works in the opposite direction:

Definition. Let $\phi : M \to N$ be smooth. Let $p \in M$ and $\eta \in T^*_{\phi(p)}(N)$. The *pull-back* of η with respect to ϕ is $\phi^*(\eta) \in T^*_p(M)$ defined by $(\phi^*(\eta))(X) = \eta(\phi_*(X))$ for any $X \in T_p(M)$.

Lemma. Let $f : N \to \mathbb{R}$. Then $\phi^*(df) = d(\phi^*(f))$.
Proof. Let $X \in T_p(M)$. Then

$$(\phi^*(df))(X) = (df)(\phi_*(X)) = (\phi_*(X))(f) = X(\phi^*(f)) = (d(\phi^*(f)))(X) \tag{222}$$

The first equality is the definition of ϕ^*, the second is the definition of df, the third is the previous Lemma and the fourth is the definition of $d(\phi^*(f))$. Since X is arbitrary, the result follows.

Exercise. Use coordinates x^μ and y^α as before. Show that the components of $\phi^*(\eta)$ are related to the components of η by

$$(\phi^*(\eta))_\mu = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \eta_\alpha \tag{223}$$

Remarks.

1. In all of the above, the point p was arbitrary so push-forward and pull-back can be applied to vector and covector *fields*, respectively.

2. The pull-back can be extended to a tensor S of type $(0, s)$ by defining $(\phi^*(S))(X_1, \ldots X_s) = S(\phi_*(X_1), \ldots \phi_*(X_n))$ where $X_1, \ldots, X_s \in T_p(M)$. Similarly, one can push-forward a tensor of type $(r, 0)$ by defining $\phi_*(T)(\eta_1, \ldots, \eta_r) = T(\phi^*(\eta_1), \ldots, \phi^*(\eta_r))$ where $\eta_1, \ldots, \eta_s \in T^*_p(M)$. The components of these tensors are given by

$$(\phi^*(S))_{\mu_1\ldots\mu_s} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right)_p \cdots \left(\frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}}\right)_p S_{\alpha_1\ldots\alpha_s} \tag{224}$$

$$(\phi_*(T))^{\alpha_1\ldots\alpha_r} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right)_p \cdots \left(\frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}}\right)_p T^{\mu_1\ldots\mu_r} \tag{225}$$

Example. The embedding of S^2 into Euclidean space. Let $M = S^2$ and $N = \mathbb{R}^3$. Define $\phi : M \to N$ as the map which sends the point on S^2 with spherical polar coordinates $x^\mu = (\theta, \phi)$ to the point $y^\alpha = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{R}^3$. Consider the Euclidean metric g on $\mathbb{R}^3$, whose components are the identity matrix $\delta_{\alpha\beta}$. Pulling this back to S^2 using (224) gives $(\phi^* g)_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$ (check!), the unit round metric on S^2.

29 Diffeomorphisms, Lie Derivative

Definition. A map $\phi : M \to N$ is a *diffeomorphism* iff it 1-1 and onto, smooth, and has a smooth inverse.

Remark. This implies that M and N have the same dimension. In fact, M and N have identical manifold structure.

With a diffeomorphism, we can extend our definitions of push-forward and pull-back so that they apply for any type of tensor:

Definition. Let $\phi : M \to N$ be a diffeomorphism and T a tensor of type (r, s) on M. Then the *push-forward* of T is a tensor $\phi_*(T)$ of type (r, s) on N defined by (for arbitrary $\eta_i \in T^*_{\phi(p)}(N)$, $X_i \in T_{\phi(p)}(N)$)

$$\phi_*(T)(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) = T(\phi^*(\eta_1), \ldots, \phi^*(\eta_r), (\phi^{-1})_*(X_1), \ldots, (\phi^{-1})_*(X_s)) \tag{226}$$

Exercises.

1. Convince yourself that push-forward commutes with the contraction and outer product operations.

2. Show that the analogue of equation (225) for a $(1, 1)$ tensor is

$$(\phi_*(T))^\mu{}_\nu = \left(\frac{\partial y^\mu}{\partial x^\rho}\right)_p \left(\frac{\partial x^\sigma}{\partial y^\nu}\right)_p T^\rho{}_\sigma \tag{227}$$

   (We don't need to use indices α, β etc because now M and N have the same dimension.) Generalize this result to a (r, s) tensor.

Remarks.

1. Pull-back can be defined in a similar way, with the result $\phi^* = (\phi^{-1})_*$.

2. We've taken an "active" point of view, regarding a diffeomorphism as a map taking a point p to a new point $\phi(p)$. However, there is an alternative "passive" point of view in which we consider a coordinate chart x^μ defined near p and another chart y^μ defined near $\phi(p)$. Regarding the coordinates y^μ as functions on N, we can pull them back to define corresponding coordinates, which we also call y^μ, on M. So now we have two coordinate systems defined near p. The components of tensors at p in the new coordinate basis are given by the tensor transformation law, which is exactly the RHS of (227).

3. In GR we describe physics with a manifold M on which certain tensor fields e.g. the metric g, Maxwell field F etc. are defined. If $\phi : M \to N$ is a diffeomorphism then there is no way of distinguishing $(M, g, F, \ldots)$ from $(N, \phi_*(g), \phi_*(F), \ldots)$; they give equivalent descriptions of physics. If we set $N = M$ this reveals that the set of tensor fields $(\phi_*(g), \phi_*(F), \ldots)$ is physically indistinguishable from $(g, F, \ldots)$. (Adopting the passive point of view, this is because they differ by a coordinate transformation.) If two sets of tensor fields are not related by a diffeomorphism then they *are* physically distinguishable. It follows that diffeomorphisms are the gauge symmetry (redundancy of description) in GR.

   This raises the following puzzle. The metric tensor is symmetric and hence has 10 independent components. The Einstein equation appears to give 10 independent equations, which looks good. But the Einstein equation should not determine the components of the metric tensor uniquely, but only up to diffeomorphisms. The resolution is that not all components of the Einstein equations are truly independent because they are related by the contracted Bianchi identity.

Note that diffeomorphisms allow us to compare tensors defined at different points via push-forward or pull-back. This leads to a notion of a tensor field possessing symmetry:

Definition. A diffeomorphism $\phi : M \to M$ is a *symmetry transformation* of a tensor field T iff $\phi_*(T) = T$ everywhere. A symmetry transformation of the metric tensor is called an *isometry*.

Definition. Let X be a vector field on a manifold M. Let ϕ_t be the map which sends a point $p \in M$ to the point parameter distance t along the integral curve of X through p (this might be defined only for small enough t). It can be shown that ϕ_t is a diffeomorphism.

Remarks.

1. Note that ϕ_0 is the identity map and $\phi_s \circ \phi_t = \phi_{s+t}$. Hence $\phi_{-t} = (\phi_t)^{-1}$. If ϕ_t is defined for all $t \in \mathbb{R}$ (in which case we say the integral curves of X are *complete*) then these diffeomorphisms form a 1-parameter abelian group.

2. Given X we've defined ϕ_t. Conversely, if one has a 1-parameter abelian group of diffeomorphisms ϕ_t (i.e. one satisfying the rules just mentioned) then through any point p one can consider the curve with parameter t given by $\phi_t(p)$. Define X to be the tangent to this curve at p. Doing this for all p defines a vector field X. The integral curves of X generate ϕ_t in the sense defined above.

3. If we use $(\phi_t)_*$ to compare tensors at different points then the parameter t controls how near the points are. In particular, in the limit $t \to 0$, we are comparing tensors at infinitesimally nearby points. This leads to the notion of a new type of derivative:

Definition. The *Lie derivative* of a tensor field T with respect to a vector field X at p is

$$(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{((\phi_{-t})_* T)_p - T_p}{t} \tag{228}$$

Remark. The Lie derivative wrt X is a map from (r, s) tensor fields to (r, s) tensor fields. It obeys $\mathcal{L}_X(\alpha S + \beta T) = \alpha \mathcal{L}_X S + \beta \mathcal{L}_X T$ where α and β are constants.

The easiest way to demonstrate other properties of the Lie derivative is to introduce coordinates in which the components of X are simple. Let Σ be a hypersurface that has the property that X is nowhere tangent to Σ (in particular $X \neq 0$ on Σ). Let x^i, $i = 1, 2, \ldots, n-1$ be coordinates on Σ. Now assign coordinates (t, x^i) to the point parameter distance t along the integral curve of X that starts at the point with coordinates x^i on Σ:

This defines a coordinate chart (t, x^i) at least for small t, i.e., in a neighbourhood of Σ. Furthermore, the integral curves of X are the curves (t, x^i) with fixed x^i and parameter t. The tangent to these curves is $\partial/\partial t$ so we have constructed coordinates such that $X = \partial/\partial t$. The diffeomorphism ϕ_t is very simple: it just sends the point with coordinates $x^\mu = (s, x^i)$ to the point with coordinates $y^\mu = (s+t, x^i)$ hence $\partial y^\mu/\partial x^\nu = \delta^\mu_\nu$. The generalization of (227) to a (r, s) tensor then gives

$$[((\phi_t)_*(T))^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_{\phi_t(p)} = [T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_p \tag{229}$$

and hence

$$[((\phi_t)_*(T))^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_p = [T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_{\phi_{-t}(p)} \tag{230}$$

It follows that, if p has coordinates (s, x^i) in this chart,

$$\begin{aligned}(\mathcal{L}_X T)^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s} &= \lim_{t\to 0}\frac{1}{t}\left(T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}(s+t, x^i) - T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}(s, x^i)\right)\\ &= \left[\frac{\partial}{\partial t}T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}(t, x^i)\right]_{(s,x^i)}\end{aligned} \tag{231}$$

So *in this chart*, the Lie derivative is simply the partial derivative with respect to the coordinate t. It follows that the Lie derivative has the following properties:

1. It obeys the Leibniz rule: $\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes \mathcal{L}_X T$.

2. It commutes with contraction.

Now let's derive a basis-independent formula for the Lie derivative. First consider a function f. In the above chart, we have $\mathcal{L}_X f = (\partial/\partial t)(f)$. However, in this chart we also have $X(f) = (\partial/\partial t)(f)$. Hence

$$\mathcal{L}_X f = X(f) \tag{232}$$

Both sides of this expression are scalars and hence this equation must be valid in any basis. Next consider a vector field Y. In our coordinates above we have

$$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} \tag{233}$$

but we also have

$$[X, Y]^\mu = \frac{\partial Y^\mu}{\partial t} \tag{234}$$

If two vectors have the same components in one basis then they are equal in all bases. Hence we have the basis-independent result

$$\mathcal{L}_X Y = [X, Y] \tag{235}$$

Remark. Let's compare the Lie derivative and the covariant derivative. The former is defined on any manifold whereas the latter requires extra structure (a connection). Equation (235) reveals that the Lie derivative wrt X at p depends on X_p *and the first derivatives of* X at p. The covariant derivative wrt X at p depends only on X_p, which enables us to remove X to define the tensor ∇T, a covariant generalization of partial differentiation. It is not possible to define a corresponding tensor $\mathcal{L}T$ using the Lie derivative. Only $\mathcal{L}_X T$ makes sense.

On examples sheet 3, you are asked to derive the formula for the Lie derivative of a covector ω valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu \tag{236}$$

and show that this can also be written in the basis-independent form (where ∇ is the Levi-Civita connection)

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b \tag{237}$$

You are also asked to show that the Lie derivative of the metric is

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho \tag{238}$$

and that this can be written in the basis-independent form

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a \tag{239}$$

Remark. If ϕ_t is a symmetry transformation of T (for all t) then $\mathcal{L}_X T = 0$. If ϕ_t are a 1-parameter group of isometries then $\mathcal{L}_X g = 0$, i.e.,

$$\nabla_a X_b + \nabla_b X_a = 0 \tag{240}$$

This is *Killing's equation* and solutions are called *Killing vector fields*. Consider the case in which there exists a chart for which the metric tensor does not depend on some coordinate z. Then equation (238) reveals that $\partial/\partial z$ is a Killing vector field. Conversely, if the metric admits a Killing vector field then equation (231) demonstrates that one can introduce coordinates such that the metric tensor is independent of one of the coordinates.

Lemma. Let X^a be a Killing vector field and let V^a be tangent to an affinely parameterized geodesic. Then $X_a V^a$ is constant along the geodesic.
Proof. The derivative of $X_a V^a$ along the geodesic is

$$\begin{aligned}\frac{d}{d\tau}(X_a V^a) = V(X_a V^a) &= \nabla_V (X_a V^a) = V^b \nabla_b (X_a V^a) \\ &= V^a V^b \nabla_b X_a + X_a V^b \nabla_b V^a \qquad (241)\end{aligned}$$

The first term vanishes because Killing's equation implies that $\nabla_b X_a$ is antisymmetric. The second term vanishes by the geodesic equation.

30 The Schwarzschild solution

This is probably the most important exact solution of the vacuum Einstein equation. It was also the first non-trivial solution to be discovered (in 1916). To a good approximation, the Sun is spherically symmetric and therefore we expect its gravitational field also to be spherically symmetric. What does this mean? You are familiar with the idea that a round sphere is invariant under rotations, which form the group $SO(3)$. In the language of the previous section, this can be phrased as follows. Note that the set of all isometries of a manifold with metric forms a group. Consider the unit round metric on S^2:

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \qquad (242)$$

The isometry group of this metric is $SO(3)$. Any 1-dimensional subgroup of $SO(3)$ gives a 1-parameter family of isometries, and hence a Killing vector field. The Killing vector fields of S^2 are explored on examples sheet 3. A *spacetime* is spherically symmetric if it possesses the same symmetries as a round S^2:

Definition. A spacetime is *spherically symmetric* if its isometry group contains an $SO(3)$ subgroup whose orbits are 2-spheres. (The *orbit* of a point p under an isometry group G is the set of points that one obtains by acting on p with an element of G.)

Remark. The statement about the orbits is important: there are examples of spacetimes with $SO(3)$ isometry group in which the orbits of $SO(3)$ are 3-dimensional (e.g. Taub-NUT space: see Hawking and Ellis).

Theorem (Birkhoff). The unique spherically symmetric solution of the vacuum Einstein equation is the *Schwarzschild solution*. In *Schwarzschild coordinates* (t, r, θ, ϕ) it has metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{243}$$

where M is a real constant and θ, ϕ parameterize S^2.
Proof. See Hawking & Ellis (rigorous) or Carroll (not so rigorous).

Remarks.

1. The word *vacuum* is important: this metric applies only *outside* any matter present in the spacetime, e.g., it applies outside the surface of the Sun.

2. The spherical symmetry is obvious: $SO(3)$ just acts on the S^2 part of the metric, leaving the t and r coordinates fixed. Surfaces of constant t and r are two-spheres of area $4\pi r^2$. This is actually how the "radial" coordinate r is defined. (Note that r is not the "distance from the origin", in fact this notion does not make sense. Can you see why?)

3. $r \to -r$ has the same effect as $M \to -M$ so wlog $r \geq 0$.

Note that if $M = 0$ then the Schwarzschild solution is just Minkowski spacetime in spherical polar coordinates. If $M \neq 0$ then for $r \gg |M|$, the metric is almost Minkowskian: the solution is *asymptotically flat*. (This term will be defined precisely in the black holes course.) So, at large r, the coordinates have essentially the same interpretation as they do in Minkowski spacetime. At large r we can approximate the metric as

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2, \tag{244}$$

where $d\Omega^2$ defined in (242) is a convenient notation for the S^2 metric. Let's define a new coordinate R by

$$r = R\sqrt{1 + \frac{2M}{R}} \tag{245}$$

Note that $r \gg |M|$ iff $R \gg |M|$, in which case $r = R + M + \mathcal{O}(M^2/R)$ and hence $dr = (1+\mathcal{O}(M^2/R^2))dR$. Plugging this into the metric and neglecting terms of order M^2/R^2 now gives

$$ds^2 \approx -\left(1 - \frac{2M}{R}\right) dt^2 + \left(1 + \frac{2M}{R}\right)\left(dR^2 + R^2 d\Omega^2\right) \tag{246}$$

But this is exactly the static weak field metric we encountered earlier, albeit with Cartesian coordinates (x, y, z) replaced by spherical polar coordinates (R, θ, ϕ). The Newtonian potential is $\Phi = -M/R$. This is the potential that arises far from an body of mass M near the origin. Hence we deduce that the parameter M in the Schwarzschild spacetime is the mass of whatever body is creating this gravitational field. Therefore we assume $M > 0$ henceforth. (The black holes course will give a more careful discussion of how to define mass in GR and the interpretation of the case $M < 0$.)

Although we assumed only spherical symmetry, the Schwarzschild solution has another symmetry: the metric is independent of t (i.e. $\partial/\partial t$ is a Killing vector field). The full isometry group is $\mathbb{R} \times SO(3)$ where $\mathbb{R}$ denotes time translations. In other words, *the gravitational field outside a spherical body is independent of time.* This is true even if the body itself is time-dependent. For example, consider a spherical star that "uses up its nuclear fuel". It will collapse under its own gravity, a time-dependent process. But the spacetime outside the star always will be described by the time-independent Schwarzschild solution.

In the coordinate basis associated with Schwarzschild coordinates, the metric $g_{\mu\nu}$ is non-degenerate except for a few special values of the coordinates. Clearly there is a problem at $\theta = 0, \pi$ but this is just the usual problem of the (θ, ϕ) coordinates not covering all of S^2. This problem can be overcome by transforming to new coordinates on S^2 e.g. the coordinates (θ', ϕ') discussed earlier. More seriously, some metric components diverge at $r = 2M$ and at $r = 0$. We shall discuss the meaning of this later. For now, note that for the Sun, $2M$ is about 3 km. The radius of the Sun is 7×10^5 km. Hence $r = 2M$ is well inside the Sun, where the Schwarzschild solution is not applicable anyway. Later, we'll show that if no matter is present then the Schwarzschild spacetime describes a black hole, with $r = 2M$ the surface of the hole.

Exercise. Let Alice and Bob be at rest in the Schwarzschild spacetime (i.e. they have worldlines with constant r, θ, ϕ). Let r_A and r_B be their radial coordinates. Repeat the gravitational redshift calculation of section 2.6 using the Schwarzschild spacetime. Show that

$$\Delta\tau_B = \left(1 - \frac{2M}{r_B}\right)^{1/2} \left(1 - \frac{2M}{r_A}\right)^{-1/2} \Delta\tau_A \tag{247}$$

Assume Bob has $r_B \gg 2M$ so the first factor can be approximated by 1. Note that $\Delta\tau_B > \Delta\tau_A$ so signals sent by Alice undergo a redshift, by a factor that diverges as $r_A \to 2M$.

31 Geodesics of the Schwarzschild solution

Exercise (examples sheet 1). Show that the t and ϕ components of the geodesic equation are

$$\frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0, \qquad \frac{d}{d\tau}\left(r^2 \sin^2\theta \frac{d\phi}{d\tau}\right) = 0, \tag{248}$$

where τ is proper time (for timelike geodesics) or an affine parameter (for null geodesics). Show that the θ component is

$$\frac{d}{d\tau}\left(r^2 \frac{d\theta}{d\tau}\right) - r^2 \sin\theta \, \cos\theta \left(\frac{d\phi}{d\tau}\right)^2 = 0. \tag{249}$$

A final equation can be obtained from $g_{ab}u^a u^b = -\sigma$, where u^a is the tangent to the geodesic and $\sigma = 1$ for timelike geodesics and $\sigma = 0$ for null geodesics:

$$-\sigma = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right], \tag{250}$$

Note that equations (248) can be integrated immediately:

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = E, \qquad r^2 \sin^2\theta \frac{d\phi}{d\tau} = h, \tag{251}$$

where E and h are constants, i.e, conserved quantities along the geodesic. The existence of these conserved quantities is a consequence of the fact that the Lagrangian from which geodesics are derived is invariant under translations of t and ϕ. More geometrically, it is because $\partial/\partial t$ and $\partial/\partial\phi$ are Killing vector fields and so give rise to conserved quantities along geodesics.

Consider a timelike geodesic which extends to $r \gg M$ so $E \approx dt/d\tau$. Since the spacetime is asymptotically flat, we can use the Minkowski spacetime result that $dt/d\tau$ is the "time" component of the 4-velocity, which is the energy per unit rest mass of the particle. Hence we shall call E the energy per unit rest mass of the particle in general. In the limit of slow motion we have $\tau \approx t$ and then we see h is the angular momentum per unit mass of the particle about the z-axis. Therefore we shall call h the angular momentum per unit rest mass more generally. In the null case, the freedom to rescale affine parameter $\tau \to a\tau$ implies that E and h do not have direct physical significance. However, the ratio h/E is invariant under this rescaling. Its physical significance is discussed below.
Eliminating $d\phi/d\tau$ from the θ equation and rearranging gives

$$r^2 \frac{d}{d\tau}\left(r^2 \frac{d\theta}{d\tau}\right) - h^2 \frac{\cos\theta}{\sin^3\theta} = 0. \tag{252}$$

As we have emphasized previously, one can define spherical polar coordinates on S^2 in many different ways. It is convenient to rotate our (θ, ϕ) coordinates so that our geodesic has $\theta = \pi/2$ and $d\theta/d\tau = 0$ at $\tau = 0$, i.e., the geodesic initially lies in, and is moving tangentially to, the "equatorial plane" $\theta = \pi/2$. We emphasize: this is just a choice of the coordinates (θ, ϕ). Now, whatever $r(\tau)$ is (and we don't know yet), equation (252) is a second order ODE for θ with initial conditions $\theta = \pi/2$, $d\theta/d\tau = 0$. One solution of this initial value problem is $\theta(\tau) = \pi/2$ for all τ. Standard uniqueness results for ODEs guarantee that this is the unique solution. Hence we have shown that we can always choose our θ, ϕ coordinates so that the geodesic is confined to the equatorial plane. We shall assume this henceforth.
Now we can eliminate $dt/d\tau$ and $d\phi/d\tau$ from equation (250) and set $\theta = \pi/2$. Rearranging gives

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{2}E^2, \tag{253}$$

where

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right) = \frac{1}{2}\sigma - \sigma\frac{M}{r} + \frac{h^2}{2r^2} - \frac{Mh^2}{r^3} \tag{254}$$

Hence the radial motion of the geodesic is the same as that of a Newtonian particle of unit mass, with total energy $E^2/2$, moving in a 1-dimensional potential $V(r)$. Of course, the same is true for particles orbiting a spherically symmetric body in Newtonian theory. The difference is in the effective potential V: the final term is absent in Newtonian theory. This term decays faster than the other terms so its effects are most pronounced at small r, i.e., near to the body creating the gravitational field.

32 Null geodesics

Consider null geodesics: $\sigma = 0$. The effective potential has the following form:

There is a maximum at $r = 3M$. Hence there is a null geodesic for which $r = 3M$, i.e., a circular orbit. However, since this corresponds to a maximum of the potential, it is unstable: a small perturbation would cause this geodesic either to fall to smaller r or escape to larger r. For the Sun, this orbit is unphysical since it lies well inside the surface, where the Schwarzschild solution is not valid.
The behaviour of null geodesics is investigated in detail on examples sheet 3. We'll summarize the results here.
Solving the geodesic equation at large r (where the metric is almost flat) reveals that a geodesic approaches a straight line, parallel to, and distance $b = |h/E|$ from, a lines of constant ϕ. This parameter b is called the *impact parameter* of the geodesic (recall that this is independent of the choice of affine parameter for the geodesic).
The qualitative property of the geodesic follows from the "particle in a potential" analogy discussed above. If $b < \sqrt{27}M$, then the "energy" of the particle exceeds the potential barrier and so a particle incident from large r will simply fall to $r = 2M$. This implies that our geodesic will spiral all the

way to $r = 2M$:

On the other hand, if $b > \sqrt{27}M$ then the particle will be reflected by the potential barrier and return to large r, where it will again approach a straight line with impact parameter b but now centered on a new value of ϕ. In flat spacetime, the change in ϕ along the geodesic is $\Delta\phi = \pi$ but in the Schwarzschild geometry, the geodesic is attracted by the gravitational field so $\Delta\phi > \pi$:

$\Delta\phi$ can be calculated for $2M/b \ll 1$ (for a light ray grazing the surface of the Sun, $2M/b \sim 10^{-5}$), with the result

$$\Delta\phi \approx \pi + \frac{4M}{b} \tag{255}$$

For a light ray grazing the surface of the Sun, $4M/b$ is about 1.7 seconds of arc. This prediction has been confirmed by observations, starting with Eddington's famous expedition in 1919.
Another important test of GR is the *Shapiro time delay*. This effect concerns a radar signal sent from the Earth to pass close to the Sun, reflect off another planet or a spacecraft, and then return to Earth. What is the time

interval measured on Earth between emission of the signal and detection of the reflected signal?
The time taken for this experiment is sufficiently small that the motion of the Earth and the reflector around the Sun can be neglected, so we treat the Earth and reflector as at rest at Schwarzschild radius r_E and r_R respectively. Let r_0 denote the minimum value of r along the null geodesic followed by the signal:

A calculation (examples sheet 3) reveals that the proper time interval measured on Earth between emission of the signal and receiving the reflected signal is

$$\Delta\tau = (\Delta\tau)_{\text{flat}} + 2M\left[\log\left(\frac{4r_E r_R}{r_0^2}\right) + 1 - \frac{r_R}{r_E}\right] \tag{256}$$

where $(\Delta\tau)_{\text{flat}}$ is the proper time taken by the same geodesic in flat spacetime. The correction term is positive (unless $r_R \gg r_E$) so GR predicts a *delay* in the time taken relative to the flat spacetime result. This effect was detected during the 1976 Viking mission to Mars. Taking r_0 to be the radius of the Sun gives a total time for the trip of about 41 minutes, and the time delay due to GR is only about 250μs. Nevertheless, the prediction of GR was confirmed. One way to parameterize the result is to replace $(1-2M/r)^{-1}$ in the Schwarzschild metric with $(1-2\gamma M/r)^{-1}$, repeat the analysis above, and then compare the result to observations to determine γ. The observations imply $\gamma = 1.000 \pm 0.002$, confirming the prediction of GR to high accuracy.

33 Timelike geodesics

Now consider timelike geodesics ($\sigma = 1$). A planet orbiting the Sun follows a timelike geodesic of the Schwarzschild solution (with small corrections coming from the influence of other planets). The effective potential has turning points where

$$r_\pm = \frac{h^2 \pm \sqrt{h^4 - 12h^2M^2}}{2M} \tag{257}$$

If $h^2 < 12M^2$ then there are no turning points, the effective potential is a monotonically increasing function of r:

A free particle incident from large r will spiral into $r = 2M$. If $h^2 > 12M^2$ then there are two turning points. $r = r_+$ is a minimum and $r = r_-$ a maximum:

Hence there exist stable circular orbits with $r = r_+$ and unstable circular orbits with $r = r_-$.

Exercise. Show that $3M < r_- < 6M < r_+$.

$r_+ = 6M$ is called the *innermost stable circular orbit* (ISCO). For the solar system, this lies will inside the Sun where the Schwarzschild solution is not valid. But for a black hole it lies outside the hole. There is no analogue of the ISCO in Newtonian theory, for which all circular orbits are stable and exist down to arbitrarily small r.

The energy per unit rest mass of a circular orbit can be calculated using $E^2/2 = V(r)$ (since $dr/d\tau = 0$). Evaluating at $r = r_\pm$ gives (exercise)

$$E = \frac{r - 2M}{r^{1/2}(r - 3M)^{1/2}} \tag{258}$$

Hence an orbit with large r has $E \approx 1 - M/(2r)$, i.e., its energy is $m - Mm/(2r)$ where m is the mass of the body. The first term is just the rest mass energy ($E = mc^2$) and the second term is the gravitational binding energy of the orbit.

Many black holes are surrounded by an *accretion disc*: a disc of material orbiting the black hole, perhaps being stripped off a nearby star by tidal forces due to the black hole's gravitational field. As a first approximation, we can treat particles in the disc as moving on geodesics. A particle in this material will gradually lose energy e.g. because of friction in the disc and so its value of E will decrease. This implies that r will decreases: the particle will gradually spiral in to smaller and smaller r. This is a very slow process so it can be approximated by the particle moving slowly from one stable circular orbit to another. Eventually the particle will reach the ISCO, which has $E = \sqrt{8/9}$, after which it falls rapidly into the hole. The energy that the particle loses in this process leaves the disc as radiation, typically X-rays. The fraction of rest mass conveted to radiation in this process is $1 - \sqrt{8/9} \approx 0.06$. This is an enormous fraction of the energy, much higher than the fraction of rest mass energy liberated in nuclear reactions. That is why black holes are believed to power some of the most energetic phenomena in the universe e.g. quasars.

Clearly there also exist non-circular bound orbits in which r oscillates around the local minimum of the potential at $r = r_+$. The *perihelion* of an orbit denotes the point on the orbit with the smallest r. In Newtonian theory, for which the r^{-3} term is absent in the effective potential, these orbits are ellipses. The change in ϕ between two successive perihelions is therefore $\Delta\phi = 2\pi$. In GR, the presence of the r^{-3} term leads to *precession* of the perihelion: $\Delta\phi > 2\pi$ (examples sheet 3). In the solar system, the effect is largest for the planet nearest the Sun, i.e., Mercury, for which the predicted precession is 42.98 seconds of arc per century. The measured precession is much larger (5599.74" $\pm$ 0.41" per century) but when one subtracts various known Newtonian effects (e.g. precession of the Earth's rotation axis, the gravitational attraction of Venus) one is left with 42.98" $\pm$ 0.04" per century. The prediction of GR is confirmed to high accuracy.

34 The Schwarzschild black hole

So far, we have used the Schwarzschild metric to describe the spacetime outside a spherical star. Let's now investigate the Schwarzschild metric as a solution that is valid everywhere. This means we need to understand what happens at $r = 2M$.

Consider the Schwarzschild solution with $r > 2M$. The analysis of geodesics reveals that some geodesics reach $r = 2M$ in finite affine parameter τ. Let's consider the simplest type of geodesic: radial null geodesics. "Radial" means that θ and ϕ are constant along the geodesic, so $h = 0$. By rescaling the affine parameter τ we can arrange that $E = 1$. The geodesic equation reduces to

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad \frac{dr}{d\tau} = \pm 1 \tag{259}$$

where the upper sign is for an outgoing geodesic (i.e. increasing r) and the lower for ingoing. From the second equation it is clear that an ingoing geodesic starting at some $r > 2M$ will reach $r = 2M$ in finite affine parameter. Along such a geodesic we have

$$\frac{dt}{dr} = -\left(1 - \frac{2M}{r}\right)^{-1} \tag{260}$$

The RHS has a simple pole at $r = 2M$ and hence t diverges logarithmically as $r \to 2M$. To investigate what is happening at $r = 2M$, define the "Regge-Wheeler radial coordinate" r_* by

$$r_* = r + 2M \log |\frac{r}{2M} - 1| \qquad \Rightarrow \qquad dr_* = \frac{dr}{\left(1 - \frac{2M}{r}\right)} \tag{261}$$

(We're interested only in $r > 2M$ for now, the modulus signs are for later use.) Note that $r_* \sim r$ for large r and $r_* \to -\infty$ as $r \to 2M$. Along a radial null geodesic we have

$$\frac{dt}{dr_*} = \pm 1 \tag{262}$$

so

$$t \mp r_* = \text{constant}. \tag{263}$$

Let's define a new coordinate v by

$$v = t + r_* \tag{264}$$

so that v is constant along ingoing radial null geodesics. Now let's use (v, r, θ, ϕ) as coordinates instead of (t, r, θ, ϕ). We eliminate t by $t = v - r_*(r)$ and hence

$$dt = dv - \frac{dr}{\left(1 - \frac{2M}{r}\right)} \tag{265}$$

Substituting this into the metric gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{266}$$

The new coordinates are called *ingoing Eddington-Finkelstein coordinates*. In these coordinates, the metric and the inverse metric both are *smooth* at $r = 2M$. The Schwarzschild spacetime can be *extended* through the surface $r = 2M$ to a new region with $r < 2M$. (The significance of the inverse metric being smooth is that it guarantees that no eigenvalue of the metric can vanish so the metric remains Lorentzian everywhere.) Note that this new region is still spherically symmetric. How is this consistent with Birkhoff's theorem?

Exercise. For $r < 2M$, define r_* by (261) and t by (264). Show that if the metric (266) is transformed to coordinates (t, r, θ, ϕ) then it becomes (243) but now with $r < 2M$.

Note that ingoing radial null geodesics in the EF coordinates have $dr/d\tau = -1$ (and constant v). Hence such geodesics will reach $r = 0$ in finite affine parameter. What happens there? A calculation (examples sheet 2) gives

$$R_{abcd} R^{abcd} \propto \frac{M^2}{r^6} \tag{267}$$

This diverges as $r \to 0$. This quantity is a scalar, and therefore diverges in all charts. Therefore there exists no chart for which the metric can be smoothly extended through $r = 0$. $r = 0$ is an example of a *curvature singularity*, where tidal forces become infinite and the known laws of physics break down. Strictly speaking, $r = 0$ is not part of the spacetime manifold because the metric is not defined there.
So far we have considered ingoing radial null geodesics, which have $v =$ constant. Now consider the outgoing geodesics, i.e., $t - r_* =$ constant. In the EF coordinates this is $v = 2r_* +$ constant, i.e.,

$$v = 2r + 4M \log \left|\frac{r}{2M} - 1\right| + \text{constant} \tag{268}$$

It is interesting to plot the radial null geodesics on a spacetime diagram. Let $t_* = v - r$ so that the ingoing radial null geodesics are straight lines at 45° in the (t_*, r) plane. This gives the *Finkelstein diagram*:

Remarks

1. Knowing the ingoing and outgoing radial null geodesics lets us draw light "cones" on this diagram. Radial timelike curves have tangent vectors that lie inside the light cone at any point.

2. The "outgoing" radial null geodesics have increasing r if $r > 2M$. But if $r < 2M$ then r decreases for *both* families of null geodesics. Both reach the curvature singularity at $r = 0$ in finite affine parameter. Since nothing can travel faster than light, the same is true for radial timelike curves. In fact one can show that r decreases along any timelike or null curve (irrespective of whether or not it is radial or geodesic) in $r < 2M$. Hence no signal can be sent from a point with $r < 2M$ to a point with $r > 2M$, in particular to a point with $r = \infty$. This is the defining property of a *black hole*: a region of an asymptotically flat spacetime from which it is impossible to send a signal to infinity.

3. $r = 2M$ is a limiting case of the outgoing radial null geodesics. By writing down the geodesic equation in EF coordinates, one can confirm that radial curves with $r = 2M$ are indeed null geodesics (we missed these above because we derived the geodesic equation in Schwarzschild coordinates). The surface $r = 2M$ which bounds the black hole is

called the *event horizon*. It acts like a one-way membrane: things can fall into the black hole but nothing can get out.

Black holes form during the process of *gravitational collapse*. A star is supported against contracting under its own gravity by pressure generated by nuclear reactions in the star's core. But eventually the star will use up its nuclear "fuel" and start to contract. Once all the nuclear fuel is spent, the only way that gravitational self-attraction can be balanced is by some *non-thermal* source of pressure (non-thermal because if we wait long enough, the star will cool). One source of such pressure arises from the Pauli principle, which makes a gas of cold fermions resist being compressed too much (this is called degeneracy pressure). For example, in a white dwarf star, the degeneracy pressure of electrons balances gravity. In a neutron star, the degeneracy pressure of neutrons balances gravity. But one can show that there is a maximum mass for such stars, of around two solar masses. If a star more massive than this undergoes gravitational collapse, then either it must shed some of its mass in a supernova, or it will undergo complete gravitational collapse to form a black hole. Gravitational collapse is easy to study if we have spherical symmetry. By continuity, points on the surface of a collapsing star will follow radial timelike curves in the Schwarzschild geometry. We can depict this on a Finkelstein diagram:

The star will collapse and form a singularity in finite proper time as measured by an observer on the star's surface. Note the behaviour of the outgoing radial null geodesics: an observer with $r > 2M$ will never see the star to collapse through $r = 2M$, instead the star will appear to freeze and quickly fade from view (due to the large redshift as $r \to 2M$). For this reason, black holes used to be called *frozen stars*.
One might suspect that small departures from spherical symmetry would

become amplified during the collapse and lead to a qualitatively different picture, e.g., an explosion of the star. However, strong evidence that black holes actually form is provided by the *singularity theorem* of Penrose. This ensures that, for small departures from spherical symmetry, gravitational collapse necessarily results in the formation of a singularity. Penrose's *cosmic censorship hypothesis* asserts that such a singularity must lie inside a black hole. There is a lot of evidence (e.g. numerical simulations) that this is indeed true, but no proof. This is probably the most important problem in mathematical relativity.

35 White holes and the Kruskal extension

We defined ingoing EF coordinates using ingoing radial null geodesics. What happens if we do the same thing with outgoing radial null geodesics? Let

$$u = t - r_* \tag{269}$$

so $u =$ constant along outgoing radial null geodesics. Now introduce *outgoing Eddington-Finkelstein* (u, r, θ, ϕ). The Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega^2 \tag{270}$$

Just as for the ingoing EF coordinates, this metric and its inverse are smooth at $r = 2M$ and can therefore be extended to a new region $r < 2M$. Once again we can define Schwarzschild coordinates in $r < 2M$ to see that the metric in this region is simply the Schwarzschild metric. There is a curvature singularity at $r = 0$. However, this $r < 2M$ region is *not the same* as the $r < 2M$ region in the ingoing EF coordinates. An easy way to see this is to look at the outgoing radial null geodesics, which we saw above have $dr/d\tau = 1$. These propagate from the curvature singularity at $r = 0$, through the surface $r = 2M$ and then extend to large r. This is impossible for $r < 2M$ region we discussed above since that region is a black hole.
The $r < 2M$ region of the outgoing EF coordinates is a *white hole*: the time-reverse of a black hole. For example, one can show that no signal can be sent from a point with $r > 2M$ to a point with $r < 2M$. Any timelike curve starting with $r < 2M$ must pass through the surface $r = 2M$ within finite proper time.

Black holes are stable objects: small perturbations of a collapsing star do not change the outcome of gravitational collapse. Time-reversal implies that white holes must be unstable objects. That is why they are not believed to be relevant for astrophysics.
We have seen that the Schwarzschild spacetime can be extended in two different ways, revealing the existence of a black hole region and a white hole region. How are these different regions related to each other? This is revealed by introducing a new set of coordinates. Start in the region $r > 2M$. Define Kruskal-Szekeres coordinates (U, V, θ, ϕ) by

$$U = -e^{-u/(4M)}, \qquad V = e^{v/(4M)}, \tag{271}$$

so $U < 0$ and $V > 0$. Note that

$$UV = -e^{r_*/(2M)} = -e^{r/(2M)} \left(\frac{r}{2M} - 1 \right) \tag{272}$$

The RHS is a monotonic function of r and hence this equation determines $r(U, V)$ uniquely. We also have

$$\frac{V}{U} = -e^{t/(2M)} \tag{273}$$

which determines $t(U, V)$ uniquely.

Exercise. Show that in Kruskal-Szekeres coordinates, the metric is

$$ds^2 = -\frac{32M^3 e^{-r(U,V)/(2M)}}{r(U,V)} dU dV + r(U,V)^2 d\Omega^2 \tag{274}$$

Hint. First transform the metric to coordinates (u, v, θ, ϕ) and then to KS coordinates.
Let us now *define* the function $r(U, V)$ for $U \geq 0$ or $V \leq 0$ by (272). This new metric and its inverse can be smoothly extended through the surfaces $U = 0$ and $V = 0$ (which correspond to $r = 2M$) to new regions with $U > 0$ or $V < 0$.
Let's consider the surface $r = 2M$. Equation (272) implies that either $U = 0$ or $V = 0$. Hence KS coordinates reveal that $r = 2M$ is actually *two* surfaces, that intersect at $U = V = 0$. Similarly, the curvature singularity at $r = 0$ corresponds to $UV = 1$, a hyperbola with two branches. This information can be summarized on a Kruskal diagram:

One should think of "time" increasing in the vertical direction on this diagram. Radial null geodesics are lines of constant U or V, i.e., lines at $45°$ to the horizontal. This diagram has four regions. Region I is the region we started in, i.e., the region $r > 2M$ of the Schwarzschild solution. Region II is the black hole that we discovered using ingoing EF coordinates (note that these coordinates cover regions I and II of the Kruskal diagram), Region III is the white hole that we discovered using outgoing EF coordinates. And region IV is an entirely new region. In this region, $r > 2M$ and so this region is again described by the Schwarzschild solution with $r > 2M$. This is a new asymptotically flat region. It is isometric to region I: the isometry is $(U, V) \to (-U, -V)$. Note that it is impossible for an observer in region I to send a signal to an observer in region IV. If they want to communicate then one or both of them will have to travel into region II (and then hit the singularity).
Note that the singularity in region II appears to the future of any point. Therefore it is not appropriate to think of the singularity as a "place" inside the black hole. It is more appropriate to think of it as a "time" at which tidal forces become infinite. The black hole region is time-dependent because, in Schwarzschild coordinates, it is r, not t that plays the role of time. This region can be thought of as describing a homogeneous but anisotropic universe approaching a "big crunch". Conversely, the white hole singularity resembles a "big bang" singularity.
Most of this diagram is unphysical. If we include a timelike worldline corresponding to the surface of a collapsing star and then replace the region to the left of this line by the spacetime corresponding to the star's interior then we get a diagram in which only regions I and II appear:

Inside the matter, we define r so that the area of each S^2 is $4\pi r^2$ and $r = 0$ is just the origin of polar coordinates, where the spacetime is smooth.

36 Linearized theory

The nonlinearity of the Einstein equation makes it very hard to solve. However, in many circumstances, gravity is not strong and spacetime can be regarded as a perturbation of Minkowski spacetime. More precisely, we assume our spacetime manifold is $M = \mathbb{R}^4$ and that there exist globally defined "almost inertial" coordinates x^μ for which the metric can be written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) \tag{275}$$

with the weakness of the gravitational field corresponding to the components of $h_{\mu\nu}$ being small compared to 1. The static weak field metric that we discussed earlier is an example of such a metric. Note that we are dealing with a situation in which we have *two* metrics defined on spacetime, namely g_{ab} and the Minkowski metric η_{ab}. The former is supposed to be the physical metric, i.e., free particles move on geodesics of g_{ab}.
To leading order in the perturbation, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{276}$$

where we define

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} \tag{277}$$

To prove this, just check that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ to linear order in the perturbation. Here, and henceforth, we shall raise and lower indices using the Minkowski metric $\eta_{\mu\nu}$. To leading order this agrees with raising and lowering with $g_{\mu\nu}$. We shall determine the Einstein equation to first order in the perturbation $h_{\mu\nu}$.
To first order, the Christoffel symbols are

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma}\right), \tag{278}$$

the Riemann tensor is (neglecting $\Gamma\Gamma$ terms since they are second order in the perturbation):

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\tau}\left(\partial_\rho\Gamma^\tau_{\nu\sigma} - \partial_\sigma\Gamma^\tau_{\nu\rho}\right) \\ &= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right) \end{aligned} \tag{279}$$

and the Ricci tensor is

$$R_{\mu\nu} = \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h, \tag{280}$$

where ∂_μ denotes $\partial/\partial x^\mu$ as usual, and

$$h = h^\mu{}_\mu \tag{281}$$

To first order, the Einstein tensor is

$$G_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho \partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial^\rho \partial_\rho h\right). \tag{282}$$

The Einstein equation equates this to $8\pi T_{\mu\nu}$ (which must therefore be assumed to be small, otherwise spacetime would not be nearly flat). It is convenient to define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}, \tag{283}$$

with inverse

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}, \qquad (\bar{h} = \bar{h}^\mu{}_\mu = -h) \tag{284}$$

The linearized Einstein equation is then (exercise)

$$-\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu} \tag{285}$$

We must now discuss the gauge symmetry present in this theory. We argued above that diffeomorphisms are gauge transformations in GR. A manifold M with metric g and energy-momentum tensor T is physically equivalent to M with metric $\phi_*(g)$ and energy momentum tensor $\phi_*(T)$ if ϕ is a diffeomorphism. Now we are restricting attention to metrics of the form (275). Hence we must consider which diffeomorphisms preserve this form. A general diffeomorphism would lead to $(\phi_*(\eta))_{\mu\nu}$ very different from $\mathrm{diag}(-1,1,1,1)$ and hence such a diffeomorphism would not preserve (275). However, if we consider a 1-parameter family of diffeomorphisms ϕ_t then ϕ_0 is the identity map, so if t is small then ϕ_t is close to the identity and hence will have a small effect, i.e., $(\phi_{t*}(\eta))_{\mu\nu}$ will be close to $\mathrm{diag}(-1,1,1,1)$ and the form (275) will be preserved. For small t, we can use the definition of the Lie derivative to deduce that, for any tensor T (the $-t$ is for notational convenience)

$$\begin{aligned}(\phi_{-t})_*(T) &= T + t\mathcal{L}_X T + \mathcal{O}(t^2) \\ &= T + \mathcal{L}_\xi T + \mathcal{O}(t^2)\end{aligned} \tag{286}$$

where X^a is the vector field that generates ϕ_t and $\xi^a = tX^a$. Note that ξ^a is small so we treat it as a first order quantity. If we apply this result to the energy-momentum tensor, evaluating in our chart x^μ, then the first term is small (by assumption) so the second term is higher order and can be neglected. Hence the energy-momentum tensor is gauge-invariant to first order. The same is true for any tensor that vanishes in the unperturbed spacetime, e.g. the Riemann tensor. However, for the metric we have

$$(\phi_{-t})_*(g) = g + \mathcal{L}_\xi g + \ldots = \eta + h + \mathcal{L}_\xi \eta + \ldots \tag{287}$$

where we have neglected $\mathcal{L}_\xi h$ because this is a higher order quantity (as ξ and h both are small). Comparing this with (275) we deduce that h and $h + \mathcal{L}_\xi \eta$ described physically equivalent metric perturbations. Therefore linearized GR has the gauge symmetry $h \to h + \mathcal{L}_\xi \eta$ for small ξ^μ. In our chart x^μ, we can use (239) to write $(\mathcal{L}_\xi \eta)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and so the gauge symmetry is

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{288}$$

There is a close analogy with electromagnetism in flat spacetime, where we can introduce an electromagnetic potential A_μ, a 4-vector obeying $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. This has the gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{289}$$

for some scalar Λ. In this case, one can choose Λ to impose the *gauge condition* $\partial^\mu A_\mu = 0$. Similarly, in linearized GR we can choose ξ_μ to impose the gauge condition

$$\partial^\nu \bar{h}_{\mu\nu} = 0. \tag{290}$$

To see this, not that under the gauge transformation (288) we have

$$\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} + \partial^\nu \partial_\nu \xi_\mu \tag{291}$$

so if we choose ξ_μ to satisfy the wave equation $\partial^\nu \partial_\nu \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$ (which we can solve using a Green function) then we reach the gauge (290). This is called variously Lorenz, de Donder, or harmonic gauge. In this gauge, the linearized Einstein equation reduces to

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{292}$$

Hence, in this gauge, each component of $\bar{h}_{\mu\nu}$ satisfies the wave equation with a source given by the energy-momentum tensor. Given appropriate boundary conditions, the solution can be determined using a Green function.

We can now justify our assertion that the static weak field metric is the unique time-independent metric produced by a weak, time-independent, non-relativistic, source. More precisely, the "non-relativistic" condition is that we can choose our "almost inertial" coordinates $x^\mu = (t, \mathbf{x})$ so that

$$T_{00} = \rho(\mathbf{x}), \qquad T_{0i} \approx 0, \qquad T_{ij} \approx 0 \tag{293}$$

The assumption that T_{0i} is negligible is the assumption that matter is moving with velocities small compared to c (note that $-T_{0i}$ is the momentum density measured by an observer at rest in these coordinates) and the assumption that T_{ij} is negligible is the assumption that stresses in matter are small compared to its energy density. (For a perfect fluid, this is the condition $p \ll \rho$ we discussed previously.) Since the source is time-independent, it is natural to expect the gravitational field also to be time independent in these coordinates, so $\partial^\rho \partial_\rho \to \nabla^2$ and (292) becomes

$$\nabla^2 \bar{h}_{00} = -16\pi\rho, \qquad \nabla^2 \bar{h}_{0i} = 0, \qquad \nabla^2 \bar{h}_{ij} = 0. \tag{294}$$

If we now define $\Phi = -(1/4)\bar{h}_{00}$ then the first of the above equations is Newton's law of gravitation (1). The unique solution of the latter two equations that is regular everywhere and decays as $\mathbf{x} \to \infty$ (we expect the metric to approach the Minkowski metric far from the source) is

$$\bar{h}_{0i} = \bar{h}_{ij} = 0. \tag{295}$$

Calculating $h_{\mu\nu}$ using (284) gives

$$h_{00} = -2\Phi, \qquad h_{0i} = 0, \qquad h_{ij} = -2\Phi\delta_{ij} \tag{296}$$

which is precisely the static weak field metric we discussed earlier.

37 Gravitational waves

In vacuum, the linearized Einstein equation reduces to the source-free wave equation:

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0 \tag{297}$$

so the theory admits gravitational wave solutions. As usual for the wave equation, we can build a general solution as a superposition of plane wave solutions. So let's look for a plane wave solution:

$$\bar{h}_{\mu\nu}(x) = \mathrm{Re}\left(H_{\mu\nu} e^{ik_\rho x^\rho}\right) \tag{298}$$

where $H_{\mu\nu}$ is a constant symmetric complex matrix describing the polarization of the wave and k^μ is the (real) wavevector. We shall suppress the Re is all subsequent equations. The wave equation reduces to

$$k_\mu k^\mu = 0 \tag{299}$$

so the wavevector k^μ must be null hence these waves propagate at the speed of light relative to the background Minkowski metric. The gauge condition (290) gives

$$k^\nu H_{\mu\nu} = 0, \tag{300}$$

i.e. the waves are transverse.

Example. As an example, consider the null vector $k^\mu = \omega(1,0,0,1)$. Then $\exp(ik_\mu x^\mu) = \exp(-i\omega(t-z))$ so this describes a wave of frequency ω propagating at the speed of light in the z-direction. The transverse condition reduces to

$$H_{\mu 0} + H_{\mu 3} = 0. \tag{301}$$

Returning to the general case, the condition (290) does not eliminate all gauge freedom. Consider a gauge transformation (288). From equation (291), we see that this preserves the gauge condition (290) if ξ_μ obeys the wave equation:

$$\partial^\nu \partial_\nu \xi_\mu = 0. \tag{302}$$

Hence there is a residual gauge freedom which we can exploit to simplify the solution. Take

$$\xi_\mu(x) = X_\mu e^{ik_\rho x^\rho} \tag{303}$$

which satisfies (302) because k_μ is null. Using

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\rho \xi_\rho \tag{304}$$

we see that the residual gauge freedom in our case is

$$H_{\mu\nu} \to H_{\mu\nu} + i\left(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho\right) \tag{305}$$

Exercise. Show that the residual gauge freedom can be used to achieve "longitudinal gauge":

$$H_{0\mu} = 0 \tag{306}$$

but this still does not determine X_μ uniquely, and the freedom remains to impose the additional "trace-free" condition

$$H^\mu{}_\mu = 0. \tag{307}$$

In this gauge, we have

$$h_{\mu\nu} = \bar{h}_{\mu\nu}. \tag{308}$$

Example. Return to our wave travelling in the z-direction. The longitudinal gauge condition combined with the transversality condition (301) gives $H_{3\mu} = 0$. Using the trace-free condition now gives

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{309}$$

where the solution is specified by the two constants H_+ and $H_\times$ corresponding to two independent polarizations. So gravitational waves are transverse and have two possible polarizations. This is one way of interpreting the statement that the gravitational field has two degrees of freedom per spacetime point.

How would one detect a gravitational wave? An observer could set up a family of test particles locally. The displacement vector S^a from the observer to any particle is governed by the geodesic deviation equation. (We are taking S^a to be infinitesimal, i.e., what we called $\delta s S^a$ previously.) So we can use this equation to predict what the observer would see. We have to be careful here. It would be natural to write out the geodesic deviation equation using the almost inertial coordinates and therby determine S^μ. But how would we relate this to observations? S^μ are the components of S^a with respect to a certain basis, so how would we determine whether the variation in S^μ arises from variation of S^a or from variation of the basis? With a bit more thought, one can make this approach work but we shall take a different approach.

Consider an observer following a geodesic in a general spacetime. Our observer will be equipped with a set of measuring rods with which to measure distances. At some point p on his worldline we could introduce a local inertial frame with spatial coordinates X, Y, Z in which the observer is at rest. Imagine that the observer sets up measuring rods of unit length pointing in the X, Y, Z directions at p. Mathematically, this defines an orthonormal basis $\{e_\alpha\}$ for $T_p(M)$ (we use α to label the basis vectors because we are using

μ for our almost inertial coordinates) where $e_0^a = u^a$ (his 4-velocity) and e_i^a ($i = 1, 2, 3$) are spacelike vectors satisfying

$$u_a e_i^a = 0, \qquad g_{ab} e_i^a e_j^b = \delta_{ij} \tag{310}$$

In Minkowski spacetime, this basis can be extended to the observer's entire worldline by taking the basis vectors to have constant components (in an inertial frame), i.e., they do not depend on proper time τ. In particular, this implies that the orthonormal basis is *non-rotating*. Since the basis vectors have constant components, they are parallelly transported along the worldline. Hence, in curved spacetime, the analogue of this is to take the basis vectors to be parallelly transported along the worldline. For u^a, this is automatic (the worldline is a geodesic). But for e_i it gives

$$u^b \nabla_b e_i^a = 0 \tag{311}$$

As we discussed previously, if the e_i^a are specified at any point p then this equation determines them uniquely along the whole worldline. Furthermore, the basis remains orthonormal because parallel transport preserves inner products (examples sheet 2). The basis just constructed is sometimes called a *parallelly transported frame*. It is the kind of basis that would be constructed by an observer freely falling and carrying a set of measuring rods. Using such a basis we can be sure that an increase in a component of S^a is really an increase in the distance to the particle in a particular direction, rather than a basis-dependent effect.

Now imagine this observer sets up a family of test particles near his worldline. The deviation vector to any infinitesimally nearby particle satisfies the geodesic deviation equation

$$u^b \nabla_b (u^c \nabla_c S_a) = R_{abcd} u^b u^c S^d \tag{312}$$

Contract with e_α^a and use the fact that the basis is parallelly transported to obtain

$$u^b \nabla_b [u^c \nabla_c (e_\alpha^a S_a)] = R_{abcd} e_\alpha^a u^b u^c S^d \tag{313}$$

Now $e_\alpha^a S_a$ is a *scalar* hence the equation reduces to

$$\frac{d^2 S_\alpha}{d\tau^2} = R_{abcd} e_\alpha^a u^b u^c e_\beta^d S^\beta \tag{314}$$

where τ is the observer's proper time and $S_\alpha = e_\alpha^a S_a$ is one of the components of S_a in the parallelly transported frame. On the RHS we've used $S^d = e_\beta^d S^\beta$.

So far, the discussion has been general but now let's specialize to our gravitational plane wave. On the RHS, R_{abcd} is a first order quantity so we only need to evaluate the other quantities to leading order, i.e., we can evaluate them as if spacetime were flat. Assume that the observer is at rest in the almost inertial coordinates. To leading order, $u^\mu = (1, 0, 0, 0)$ hence

$$\frac{d^2 S_\alpha}{d\tau^2} \approx R_{\mu 00\nu} e_\alpha^\mu e_\beta^\nu S^\beta \tag{315}$$

Using the formula for the perturbed Riemann tensor (279) and $h_{0\mu} = 0$ we obtain

$$\frac{d^2 S_\alpha}{d\tau^2} \approx \frac{1}{2} \frac{\partial^2 h_{\mu\nu}}{\partial t^2} e_\alpha^\mu e_\beta^\nu S^\beta \tag{316}$$

In Minkowski spacetime we could take e_i^a aligned with the x, y, z axes respectively, i.e., $e_1^\mu = (0, 1, 0, 0)$, $e_2^\mu = (0, 0, 1, 0)$ and $e_3^\mu = (0, 0, 0, 1)$. We can use the same results here because we only need to evaluate e_α^μ to leading order. Using $h_{0\mu} = h_{3\mu} = 0$ we then have

$$\frac{d^2 S_0}{d\tau^2} = \frac{d^2 S_3}{d\tau^2} = 0 \tag{317}$$

to this order of approximation. Hence the observer sees no relative acceleration of the test particles in the z-direction, i.e, the direction of propagation of the wave. Let the observer set up initial conditions so that S_0 and its first derivatives vanish at $\tau = 0$. Then S_0 will vanish for all time. If the derivative of S_3 vanishes initially then S_3 will be constant. The same is not true for the other components.

We can choose our almost inertial coordinates so that the observer has coordinates $x^\mu \approx (\tau, 0, 0, 0)$ (i.e. $t = \tau$ to leading order along the observer's worldline). For a + polarized wave we then have

$$\frac{d^2 S_1}{d\tau^2} = \frac{1}{2}\omega^2 |H_+| \cos(\omega\tau - \alpha) S_1, \qquad \frac{d^2 S_2}{d\tau^2} = -\frac{1}{2}\omega^2 |H_+| \cos(\omega\tau - \alpha) S_2 \tag{318}$$

where we have replaced t by τ in $\partial^2 h_{\mu\nu}/\partial t^2$ and $\alpha = \arg H_+$. Since H_+ is small we can solve this perturbatively: the leading order solution is $S_1 = \bar{S}_1$, a constant (assuming that we set up initial condition so that the particles are at rest to leading order). Similarly $S_2 = \bar{S}_2$. Now we can plug these leading order solutions into the RHS of the above equations and integrate to determine the solution up to first order (again choosing constants of integration so that the particles would be at rest in the absense of the wave)

$$S_1(\tau) \approx \bar{S}_1 \left(1 - \frac{1}{2}|H_+| \cos(\omega\tau - \alpha)\right), \quad S_2(\tau) \approx \bar{S}_2 \left(1 + \frac{1}{2}|H_+| \cos(\omega\tau - \alpha)\right) \tag{319}$$

This reveals that particles are displaced outwards in the x-direction whilst being displaced inwards in the y-direction, and vice-versa. $\bar{S}_1$ and $\bar{S}_2$ give the average displacement. If the particles are arranged in the xy plane with $\bar{S}_1^2 + \bar{S}_2^2 = R^2$ then they form a circle of radius R when $\omega\tau = \alpha + \pi/2$. This will be deformed into an ellipse, then back to a circle, then an ellipse again:

Exercise. Show that the corresponding result for a $\times$ polarized wave is the same, just rotated through 45°:

There is an ongoing experiment effort to detect gravitational waves. The effect just mentioned is the basis for detection efforts. For example, the LIGO observatory has two perpendicular tunnels, each 4 km long. There are test masses (analogous to the particles above) at the end of each arm (tunnel) and where the arms meet. A beam splitter is attached to the test mass where the arms meet. A laser signal is split and sent down each arm, where it reflects off mirrors attached to the test masses at the ends of the arms. The signals are recombined and interferometry used to detect whether there has been any change in the length difference of the two arms. The effect that is being looked for is tiny: plausible sources of gravitational waves give $H_+, H_\times \sim 10^{-21}$ so the relative length change of each arm is $\delta L/L \sim 10^{-21}$.

38 The quadrupole formula

Let's return to the linearized Einstein equation with matter:

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{320}$$

Since each component of $\bar{h}_{\mu\nu}$ satisfies the inhomogeneous wave equation, the solution can be solved using the same retarded Green function that one uses in electromagnetism:

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int d^3x' \frac{T_{\mu\nu}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad t' = t - |\mathbf{x} - \mathbf{x}'| \tag{321}$$

where $|\mathbf{x} - \mathbf{x}'|$ is calculated using the Euclidean metric. In what follows, we shall consider only the spatial components of $\bar{h}_{\mu\nu}$, i.e., $\bar{h}_{ij}$. Other components can be obtained from the gauge condition (290), which gives

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}, \qquad \partial_0 \bar{h}_{00} = \partial_j \bar{h}_{0j} \tag{322}$$

Given $\bar{h}_{ij}$, the first equation can be integrated to determine $\bar{h}_{0i}$ and the second can then be integrated to determine $\bar{h}_{00}$. Assume that the matter is confined to a compact region near the origin. Then, far from the source we have $|\mathbf{x}'| \ll |\mathbf{x}| = r$ and hence

$$\bar{h}_{ij}(t, \mathbf{x}) \approx \frac{4}{r} \int d^3x' \, T_{ij}(t', \mathbf{x}') \qquad t' = t - r \tag{323}$$

The integral can be evaluated as follows. Since the matter is compactly supported, we can freely integrate by parts and discard surface terms. We can also use energy-momentum conservation, which to this order is just $\partial_\nu T^{\mu\nu} = 0$. Let's drop the primes on the coordinates in the integral for now.

$$\int d^3x \, T^{ij} \; = \; \int d^3x \, \left[\partial_k (T^{ik} x^j) - (\partial_k T^{ik}) x^j\right]$$

$$
\begin{aligned}
&= -\int d^3x\,(\partial_k T^{ik})x^j \qquad \text{integration by parts} \\
&= \int d^3x(\partial_0 T^{i0})x^j \qquad \text{conservation law} \\
&= \partial_0 \int d^3x\, T^{0i} x^j
\end{aligned} \tag{324}
$$

We can now symmetrize this equation on ij to get

$$
\begin{aligned}
\int d^3x\, T^{ij} &= \partial_0 \int d^3x\, T^{0(i}x^{j)} \\
&= \partial_0 \int d^3x\, \left[\frac{1}{2}\partial_k \left(T^{0k}x^i x^j\right) - \frac{1}{2}(\partial_k T^{0k})x^i x^j\right] \\
&= -\frac{1}{2}\partial_0 \int d^3x\,(\partial_k T^{0k})x^i x^j \qquad \text{integration by parts} \\
&= \frac{1}{2}\partial_0 \int d^3x\,(\partial_0 T^{00})x^i x^j \qquad \text{conservation law} \\
&= \frac{1}{2}\partial_0\partial_0 \int d^3x\, T^{00}x^i x^j \\
&= \frac{1}{2}\ddot{I}_{ij}(t)
\end{aligned} \tag{325}
$$

where

$$
I_{ij}(t) = \int d^3x T_{00}(t,\mathbf{x})x^i x^j \tag{326}
$$

(Note that $T_{00} = T^{00}$ and $T_{ij} = T^{ij}$ to leading order.) Hence we have

$$
\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{2}{r}\ddot{I}_{ij}(t-r) \tag{327}
$$

I_{ij} is the second moment of the energy density. It is a tensor in the Cartesian sense, i.e., it transforms in the usual way under rotations of the coordinates x^i. (The zeroth moment is the total energy in matter $\int d^3x T_{00}$, the first moment is the energy dipole $\int d^3x T_{00} x^i$.) It is closely related to the energy quadrupole tensor, which is the traceless part of I_{ij}

$$
Q_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij} \tag{328}
$$

We see that the gravitational waves arise when I_{ij} varies in time. Gravitational waves carry energy away from the souce. Calculating this is subtle

because we have seen that there is no local notion of energy density for the gravitational field. However, one can define a notion of *total* energy (of matter and the gravitational field) for an asymptotically flat spacetime. Moreover, one can calculate the rate of change of this energy at infinity. It turns out to be negative. The interpretation is that energy is carried away by gravitational waves. The rate of decrease can be interpreted as the power radiated in gravitational waves.
The qualitative form of the expression for this power can be understood as follows. By analogy with electromagnetism, we would expect the power radiated across a large sphere of radius r to be quadratic in first time derivatives of the field $\bar{h}_{\mu\nu}$ at radius r. Hence it will be quadratic in the third time derivative of I_{ij}. The power should be a scalar (in the Cartesian sense) and therefore must have the form

$$P = \alpha \, \dddot{I}_{ij} \dddot{I}_{ij} + \beta (\dddot{I}_{kk})^2 \tag{329}$$

We can appeal to Birkhoff's theorem, which tells us that there can be no gravitational radiation if the source is spherically symmetric. A spherically symmetric source has $I_{ij} \propto \delta_{ij}$. In this case, we should find $P = 0$. This implies $3\alpha + 9\beta = 0$. Our formula reduces to $P = \alpha \, \dddot{Q}_{ij} \dddot{Q}_{ij}$. A more careful determines the coefficient, with the result

$$P(t) = \frac{1}{5} \left(\dddot{Q}_{ij} \dddot{Q}_{ij} \right)_{t-r} \tag{330}$$

This is the *quadrupole formula* for energy loss via gravitational wave emission. Consider an asymmetric body of mass M and characteristic size R rotating with angular velocity Ω. On dimensional grounds we have $Q_{ij} \sim MR^2$ and $\dddot{Q}_{ij} \sim MR^2\Omega^3$. Hence we can estimate the power emitted in gravitational waves as $P \sim M^2R^4\Omega^6$. Reinstating factors of G and c, this is $GM^2R^4\Omega^6/c^5$. As the Earth orbits the Sun, it emits gravitational waves but the power emitted is tiny: about 200 watts. However, for a binary system (a pair of stars or black holes orbiting their common centre of mass), it is much larger. If the objects in the binary are very compact (e.g. neutron stars or black holes) and the radius of the orbit is small (so Ω is large) then the power emitted in gravitational radiation can be comparable to the power emitted by the Sun in electromagnetic radiation.
The quadrupole formula has been verified experimentally. In 1974, Hulse and Taylor identifed a *binary pulsar*. This is a neutron star binary in which one of

the stars is a pulsar, i.e., it emits a beam of radio waves in a certain direction. The star is rotating very rapidly and the beam (which is not aligned with the rotation axis) periodically points in our direction. Hence we receive pulses of radiation from the star. The period between successive pulses (about 0.05s) is very stable and has been measured to very high accuracy. Therefore it acts like a clock that we can observe from Earth. Using this clock we can determine the orbital period (about 7.75h) of the binary system, again with good accuracy. The system loses energy through emission of gravitational waves. This makes the period of the orbit decrease by about 10μs per year. This small effect has been measured and the result confirms the quadrupole formula to an accuracy of 0.3% (the accuracy increases the longer the system is observed). This is very strong *indirect* evidence for the existence of gravitational waves, for which Hulse and Taylor received the Nobel Prize in 1993.

39 Integration on manifolds

The Einstein equation can be obtained by extremizing an action. To understand this, we first need to discuss how to integrate on a manifold M. The best way of doing this is to introduce differential forms but for our purposes this is not necessary.
Let $\phi : \mathcal{O} \to \mathcal{U}$ be a chart with coordinates x^μ and $f : M \to \mathbb{R}$. How would we define the integral of f over $\mathcal{O}$? One definition would be

$$\int_{\mathcal{U}} d^n x f(x) \tag{331}$$

where, as usual, we are writing $f(x)$ as an abbreviation for $f(\phi^{-1}(x))$. However, this does not work because if $\phi' : \mathcal{O} \to \mathcal{U}'$ is another chart, with coordinates x'^μ then we would have

$$\int_{\mathcal{U}'} d^n x' f(x') = \int_{\mathcal{U}} d^n x ||J|| f(x), \tag{332}$$

where $||J||$ is the Jacobian:

$$||J|| = \det J^\mu{}_\nu, \qquad J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{333}$$

Since these integrals are not equal, our definition is chart-dependent, which is unsatisfactory. There is a simple solution to this problem if our manifold has a metric. The tensor transformation law gives

$$g'_{\mu\nu}(x') = (J^{-1})^{\rho}{}_{\mu}(J^{-1})^{\sigma}{}_{\nu}\, g_{\rho\sigma}(x) \tag{334}$$

hence if we define the (non-scalar) quantity

$$g = \det g_{\mu\nu} \tag{335}$$

then

$$g' = ||J||^{-2} g \qquad \Rightarrow \qquad \sqrt{|g'|} = ||J||^{-1}\sqrt{|g|} \tag{336}$$

Hence we have

$$d^n x' \sqrt{|g'|} = d^n x \sqrt{|g|} \tag{337}$$

It follows that the definition

$$\int_{\mathcal{O}} f \equiv \int_{\mathcal{U}} d^n x \sqrt{|g|} f(x) \tag{338}$$

is chart-independent so this is the definition we shall adopt. Actually, there is a subtlety: the above works only if $||J||$ is positive.

Definition. An *oriented* manifold is one with an atlas for which $||J||$ is positive for all pairs of coordinate charts.

Familiar examples such as $\mathbb{R}^n$ or S^n are oriented. We assume that M is oriented henceforth.

How do we extend our definition to all of M? Skipping some technical details, the idea is to choose an atlas with charts $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ and a "partition of unity", i.e., a set of functions $h_\alpha : M \to [0,1]$ such that $h_\alpha(p) = 0$ if $p \notin \mathcal{O}_\alpha$, and $\sum_\alpha h_\alpha(p) = 1$ for all p. We then define

$$\int_M f \equiv \sum_\alpha \int_{\mathcal{O}_\alpha} h_\alpha f \tag{339}$$

It can be shown that this definition is independent of the choice of atlas and partition of unity.

From now on, we shall adopt the following common notation

$$\int_M d^n x \, \sqrt{|g|}\, f \equiv \int_M f \tag{340}$$

This is an abuse of notation because the LHS refers to coordinates x^μ but M might not be covered by a single chart. It has the advantage of making manifest the fact that our integrals depend on the metric tensor.

Definition. The *volume* of M is obtained by setting $f = 1$: $V = \int_M d^n x \sqrt{|g|}$.

We shall need the generalization of the divergence theorem to an arbitrary manifold. To do this, we need to introduce the notion of a manifold with boundary:

Definition. A *manifold with boundary* M is defined in the same way as a manifold except that charts are maps $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where now $\mathcal{U}_\alpha$ is an open subset of $\frac{1}{2}\mathbb{R}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 \leq 0\}$. The boundary of M, denoted ∂M, is the set of points for which $x^1 = 0$. This is a manifold of dimension $n - 1$ with coordinate charts $(x^2, \ldots x^n)$. It is oriented if M is oriented.

Consider a curve in ∂M with parameter t and tangent vector X. Then $x^1(t) = 0$ so

$$dx^1(X) = X(x^1) = \frac{dx^1}{dt} = 0. \tag{341}$$

Hence $dx^1(X)$ vanishes for any X tangent to ∂M so dx^1 is normal to ∂M. Any other normal to ∂M will be proportional to dx^1. We can construct a unit normal by dividing by the norm of dx^1:

$$n_a = \frac{\pm (dx^1)_a}{\sqrt{\pm g^{bc}(dx^1)_b (dx^1)_c}} \qquad \Rightarrow \qquad g^{ab} n_a n_b = \pm 1 \tag{342}$$

One can show that this is chart independent. Here we choose the + sign if dx^1 is spacelike and the $-$ sign if dx^1 is timelike (+ if the metric is Riemannian). The sign in the numerator is in order to get the correct sign in the divergence theorem. If dx^1 is null anywhere on ∂M then one has to express the divergence theorem differently (using differential forms).
There is an obvious map $\partial M \to M$ in which we send a point in ∂M to itself. We can use this map to pull-back the metric on M to define a metric on ∂M. We can then define integration on ∂M using this metric. Finally we can state the divergence theorem:

$$\int_M d^n x \sqrt{|g|}\, \nabla_a X^a = \int_{\partial M} d^{n-1} x \sqrt{|h|}\, n_a X^a \tag{343}$$

where X^a is a vector field on M, ∇ is the Levi-Civita connection, and h denotes the determinant of the metric on ∂M. $n_a X^a$ is a scalar in M so it can be pulled back to ∂M, this is the integrand on the RHS.

40 Scalar field action

You are familiar with the idea that the equation of motion of a point particle can be obtained by extremizing an action. You may also know that the same is true for fields in Minkowski spacetime. The same is true in GR. To see how this works, consider first a scalar field, i.e., a function $\Phi : M \to \mathbb{R}$ and define the *action* as the functional

$$S[\Phi] = \int_M d^4x \sqrt{-g} L \tag{344}$$

where L is the *Lagrangian*:

$$L = -\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - V(\Phi) \tag{345}$$

and $V(\Phi)$ is called the scalar potential. Now consider a variation $\Phi \to \Phi + \delta\Phi$ for some function $\delta\Phi$ that vanishes on ∂M (in an asymptotically flat spacetime, ∂M will be "at infinity"). The change in the action is (working to linear order in $\delta\Phi$)

$$\begin{aligned}
\delta S &= S[\Phi + \delta\Phi] - S[\Phi] \\
&= \int_M d^4x \sqrt{-g} \left(-g^{ab} \nabla_a \Phi \nabla_b \delta\Phi - V'(\Phi)\delta\Phi \right) \\
&= \int_M d^4x \sqrt{-g} \left[-\nabla_a \left(\delta\Phi \nabla^a \Phi \right) + \delta\Phi \nabla^a \nabla_a \Phi - V'(\Phi)\delta\Phi \right] \\
&= \int_{\partial M} d^3x \sqrt{|h|}\, \delta\Phi\, n_a \nabla^a \Phi + \int_M d^4x \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \delta\Phi \\
&= \int_M d^4x \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \delta\Phi
\end{aligned} \tag{346}$$

Note that we have used the divergence theorem to "integrate by parts". An alternative way of writing the final expression is

$$\frac{\delta S}{\delta\Phi} = \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \tag{347}$$

Demanding that δS vanishes for arbitrary $\delta\Phi$ gives us the equation of motion $\delta S / \delta\Phi = 0$, i.e.,

$$\nabla^a \nabla_a \Phi - V'(\Phi) = 0. \tag{348}$$

The particular choice $V(\Phi) = \frac{1}{2} m^2 \Phi^2$ gives the Klein-Gordon equation.

41 The Einstein-Hilbert action

For the gravitational field, we seek an action of the form

$$S[g] = \int_M d^4x\sqrt{-g}L \tag{349}$$

where L is a scalar constructed from the metric. An obvious choice for the Lagrangian is $L \propto R$. This gives the *Einstein-Hilbert action*

$$S_{EH}[g] = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R \tag{350}$$

where the prefactor is included for later convenience. The idea is that we regard our manifold M as fixed (e.g. $\mathbb{R}^4$) and g_{ab} is determined by extremizing $S[g]$. In other words, we consider two metrics g_{ab} and $g_{ab} + \delta g_{ab}$ and demand that $S[g + \delta g] - S[g]$ should vanish to linear order in δg_{ab}. Note that δg_{ab} is the difference of two metrics and hence is a tensor field.
We need to work out what happens to $\sqrt{-g}$ and R when we vary $g_{\mu\nu}$. Recall the formula for the determinant "expanding along the μth row":

$$g = \sum_\nu g_{\mu\nu}\Delta^{\mu\nu} \tag{351}$$

where we are suspending the summation convention, μ is any fixed value, and $\Delta^{\mu\nu}$ is the cofactor matrix, whose $\mu\nu$ element is $(-1)^{\mu+\nu}$ times the determinant of the matrix obtained by deleting row μ and column ν from the metric. Note that $\Delta^{\mu\nu}$ is independent of the $\mu\nu$ element of the metric. Hence

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} = gg^{\mu\nu} \tag{352}$$

where on the RHS we recall the formula for the inverse matrix $g^{\mu\nu}$ in terms of the cofactor matrix. We can use this formula to determine how g varies under a small change $\delta g_{\mu\nu}$ in $g_{\mu\nu}$ (reinstating the summation convention):

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = gg^{\mu\nu}\delta g_{\mu\nu} = gg^{ab}\delta g_{ab} \tag{353}$$

(we can use abstract indices in the final equality since $g^{ab}\delta g_{ab}$ is a scalar) and hence

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{ab}\delta g_{ab} \tag{354}$$

Next we need to evaluate δR. To this end, consider first the change in the Christoffel symbols. $\delta\Gamma^{\mu}_{\nu\rho}$ is the difference between the components of two connections (i.e. the Levi-Civita connections associated to $g_{ab} + \delta g_{ab}$ and g_{ab}). Since the difference of two connections is a tensor, it follows that $\delta\Gamma^{\mu}_{\nu\rho}$ are components of a tensor $\delta\Gamma^{a}_{bc}$. These components are easy to evaluate if we introduce normal coordinates at p for the unperturbed connection: at p we have $g_{\mu\nu,\rho} = 0$ and $\Gamma^{\mu}_{\nu\rho} = 0$. For the perturbed connection we therefore have, at p, (to linear order)

$$\begin{aligned} \delta\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} \right) \\ &= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma} \right) \end{aligned} \tag{355}$$

In the second equality, the semi-colon denotes a covariant derivative with respect to the Levi-Civita connection associated to g_{ab}. The two expressions are equal because $\Gamma(p) = 0$. The LHS and RHS are tensors so this is a basis independent result hence we can use abstract indices:

$$\delta\Gamma^{a}_{bc} = \frac{1}{2} g^{ad} \left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d} \right) \tag{356}$$

p is arbitrary so this result holds everywhere.

Now consider the variation of the Riemann tensor. Again it is convenient to use normal coordinates at p, so at p we have (using $\delta(\Gamma\Gamma) \sim \Gamma\delta\Gamma = 0$ at p)

$$\begin{aligned} \delta R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}_{\nu\rho} \\ &= \nabla_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}_{\nu\rho} \end{aligned} \tag{357}$$

where ∇ is the Levi-Civita connection of g_{ab}. Once again we can immediately replace the basis indices by abstract indices:

$$\delta R^{a}{}_{bcd} = \nabla_c \delta\Gamma^{a}_{bd} - \nabla_d \delta\Gamma^{a}_{bc} \tag{358}$$

and p is arbitrary so the result holds everywhere. Contracting gives the variation of the Ricci tensor:

$$\delta R_{ab} = \nabla_c \delta\Gamma^{c}_{ab} - \nabla_b \delta\Gamma^{c}_{ac} \tag{359}$$

Finally we have

$$\delta R = \delta(g^{ab} R_{ab}) = g^{ab} \delta R_{ab} + \delta g^{ab} R_{ab} \tag{360}$$

where δg^{ab} is the variation in g^{ab} (*not* the result of raising indices on δg_{ab}). Using $\delta(g_{\mu\rho}g^{\rho\nu}) = \delta(\delta_\mu^\nu) = 0$ it is easy to show (exercise)

$$\delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd} \tag{361}$$

Putting everything together, we have

$$\begin{aligned} \delta R &= -g^{ac}g^{bd}R_{ab}\delta g_{cd} + g^{ab}\left(\nabla_c \delta\Gamma^c_{\ ab} - \nabla_b \delta\Gamma^c_{\ ac}\right) \\ &= -R^{ab}\delta g_{ab} + \nabla_c(g^{ab}\delta\Gamma^c_{\ ab}) - \nabla_b(g^{ab}\delta\Gamma^c_{\ ac}) \\ &= -R^{ab}\delta g_{ab} + \nabla_a X^a \end{aligned} \tag{362}$$

where

$$X^a = g^{bc}\delta\Gamma^a_{\ bc} - g^{ab}\delta\Gamma^c_{\ bc} \tag{363}$$

Hence the variation of the Einstein-Hilbert action is

$$\begin{aligned} \delta S_{EH} &= \frac{1}{16\pi}\int_M d^4x\,\delta(\sqrt{-g}\,R) \\ &= \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left(\frac{1}{2}Rg^{ab}\delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a X^a\right) \end{aligned} \tag{364}$$

The final term can be converted to a surface term on ∂M using the divergence theorem. If we assume that δg_{ab} has support in a compact region that doesn't intersect ∂M then this term will vanish (because vanishing of δg_{ab} and its derivative on ∂M implies that X^a will vanish on ∂M). Hence we have

$$\delta S_{EH} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left(-G^{ab}\right)\delta g_{ab} \tag{365}$$

where G_{ab} is the Einstein tensor. Equivalently:

$$\frac{\delta S_{EH}}{\delta g_{ab}} = -\frac{1}{16\pi}\sqrt{-g}G^{ab} \tag{366}$$

Hence extremization of S_{EH} reproduces the vacuum Einstein equation.

Exercise. Show that the vacuum Einstein equation with cosmological constant is obtained by extremizing

$$S_{EH\Lambda} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left(R - 2\Lambda\right) \tag{367}$$

Lovelock showed that the Lagrangian here is the most general scalar that can be constructed from the metric and at most two derivatives.

42 Energy momentum tensor

Next we consider the action for matter. We assume that this is given in terms of the integral of a scalar Lagrangian:

$$S_{\text{matter}} = \int d^4x\sqrt{-g}L_{\text{matter}} \tag{368}$$

here L_{matter} is a function of the matter fields (assumed to be tensor fields), their derivatives, the metric and its derivatives. An example is given by the scalar field Lagrangian discussed above. We define the energy momentum tensor by

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{ab}} \tag{369}$$

in other words, under a variation in g_{ab} we have (after integrating by parts using the divergence theorem to eliminate derivatives of δg_{ab} if present)

$$\delta S_{\text{matter}} = \frac{1}{2}\int_M d^4x\sqrt{-g}T^{ab}\delta g_{ab} \tag{370}$$

This definition clearly makes T^{ab} symmetric.

Example. Consider the scalar field action we discussed previously. Using the results for $\delta\sqrt{-g}$ and δg^{ab} derived above we have, under a variation of g_{ab}:

$$\delta S = \int_M d^4x\sqrt{-g}\left[\frac{1}{2}\nabla^a\Phi\nabla^b\Phi + \frac{1}{2}\left(-\frac{1}{2}g^{cd}\nabla_c\Phi\nabla_d\Phi - V(\Phi)\right)g^{ab}\right]\delta g_{ab} \tag{371}$$

Hence

$$T^{ab} = \nabla^a\Phi\nabla^b\Phi + \left(-\frac{1}{2}g^{cd}\nabla_c\Phi\nabla_d\Phi - V(\Phi)\right)g^{ab} \tag{372}$$

If we define the total action to be $S_{EH} + S_{\text{matter}}$ then under a variation of g_{ab} we have

$$\frac{\delta}{\delta g_{ab}}\left(S_{EH} + S_{\text{matter}}\right) = \sqrt{-g}\left(-\frac{1}{16\pi}G^{ab} + \frac{1}{2}T^{ab}\right) \tag{373}$$

and hence demanding that $S_{EH} + S_{\text{matter}}$ be extremized under variation of the metric gives the Einstein equation

$$G_{ab} = 8\pi T_{ab} \tag{374}$$

How do we know that our definition of T_{ab} gives a conserved tensor? It follows from the fact that $S_{\rm matter}$ is diffeomorphism invariant. In more detail, diffeomorphisms are a gauge symmetry so the total action $S = S_{EH} + S_{\rm matter}$ should be diffeomorphism invariant in the sense that $S[g, \Phi] = S[\phi_*(g), \phi_*(\Phi)]$ where Φ denotes the matter fields and ϕ is a diffeomorphism. The Einstein-Hilbert action alone is diffeomorphism invariant (because a diffeomorphism has the same effect as a change of coordinates, and we've defined integration of a scalar on M in a manifestly coordinate-independent way). Hence $S_{\rm matter}$ also must be diffeomorphism invariant. The easiest way of ensuring this is to take it to be the integral of a scalar Lagrangian as we assumed above. Now consider the effect of an infinitesimal diffeomorphism. As we saw when discussing linearized theory (eq (287)), an infinitesimal diffeomorphism shifts g_{ab} by

$$\delta g_{ab} = \mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \tag{375}$$

Matter fields also transform according to the Lie derivative (eq (286)), e.g., for a scalar field:

$$\delta \Phi = \mathcal{L}_\xi \Phi = \xi^a \nabla_a \Phi \tag{376}$$

Let's consider this scalar field case in detail. Assume that the matter Lagrangian is an arbitrary scalar constructed from Φ, the metric, and arbitrarily many of their derivatives (e.g. there could be a term of the form $\nabla_a \nabla_b \Phi \nabla^a \nabla^b \Phi$ or $R\Phi^2$). Under an infinitesimal diffeomorphism,

$$\begin{aligned} \delta S_{\rm matter} &= \int_M d^4x \left(\frac{\delta S_{\rm matter}}{\delta \Phi} \delta \Phi + \frac{\delta S_{\rm matter}}{\delta g_{ab}} \delta g_{ab} \right) \\ &= \int_M d^4x \left(\frac{\delta S_{\rm matter}}{\delta \Phi} \xi^b \nabla_b \Phi + \frac{1}{2}\sqrt{-g} T^{ab} \delta g_{ab} \right) \end{aligned} \tag{377}$$

The second term can be written

$$\begin{aligned} \int_M d^4x \sqrt{-g} T^{ab} \nabla_a \xi_b &= \int_M d^4x \sqrt{-g} \left[\nabla_a \left(T^{ab} \xi_b \right) - \left(\nabla_a T^{ab} \right) \xi_b \right] \\ &= -\int_M d^4x \sqrt{-g} \left(\nabla_a T^{ab} \right) \xi_b \end{aligned} \tag{378}$$

where we assume that ξ_b vanishes on ∂M so the total derivative can be discarded. Now diffeomorphism invariance implies that $\delta S_{\rm matter}$ must vanish for arbitrary ξ_b. Hence we must have

$$\frac{\delta S_{\rm matter}}{\delta \Phi} \nabla^b \Phi - \sqrt{-g} \nabla_a T^{ab} = 0. \tag{379}$$

Hence we see that if the scalar field equation of motion ($\delta S_{\rm matter}/\delta\Phi = 0$) is satisfied then

$$\nabla_a T^{ab} = 0. \tag{380}$$

This is a special case of a very general result. Diffeomorphism invariance plus the equations of motion for the matter fields implies energy-momentum tensor conservation. It applies for a matter Lagrangian constructed from tensor fields of any type (the matter fields), the metric, and arbitrarily many derivatives of the matter fields and metric.

An identical argument applied to the Einstein-Hilbert action leads to the contracted Bianchi identity (exercise):

$$\nabla_a G^{ab} = 0. \tag{381}$$

Hence the contracted Bianchi identity is equivalent to diffeomorphism invariance of the Einstein-Hilbert action.

43 Spaces of constant curvature

Definition. A manifold M with metric g has *constant curvature* if

$$R_{abcd} = 2K g_{a[c}g_{d]b} \qquad K = \text{constant} \tag{382}$$

Examples.

1. S^n with the round metric of radius $a > 0$ is the subset of $\mathbb{R}^{n+1}$ given by

$$(x^1)^2 + \ldots + (x^{n+1})^2 = a^2 \tag{383}$$

   and the metric is the one induced by pulling back the Euclidean metric on $\mathbb{R}^{n+1}$. This satisfies (382) with $K = 1/a^2$. In detail, consider $n = 3$ and paramerize S^3 by

$$\begin{aligned} x^1 &= a\sin\chi\sin\theta\cos\phi, \qquad x^2 = a\sin\chi\sin\theta\sin\phi \\ x^3 &= a\sin\chi\cos\theta, \qquad x^4 = a\cos\chi \end{aligned} \tag{384}$$

   where $0 < \chi, \theta < \pi$, $0 < \phi < 2\pi$. Pulling back the Euclidean metric using (224) gives the round metric of radius a on S^3:

$$ds^2 = a^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{385}$$

These coordinates do not cover all of S^3 but, just as for S^2, one can introduce another similar chart to cover the places where χ or θ is 0 or π or ϕ is 0 or 2π.

2. Consider $\mathbb{R}^{n+1}$ with Minkowski metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \dots (dx^n)^2 \tag{386}$$

Now consider the surface ($a > 0$)

$$-(x^0)^2 + (x^1)^2 + \dots (x^n)^2 = -a^2 \tag{387}$$

This is a double sheeted hyperboloid:

Consider one sheet of this hyperboloid, which as a manifold is just $\mathbb{R}^n$, and the metric induced by pulling back the Minkowski metric. This gives a Riemannian metric called the hyperbolic metric of radius a. $\mathbb{R}^n$ with this metric is called *hyperbolic space* H^n. This satisfies (382) with $K = -1/a^2$. In detail, for $n = 3$ parameterize H^3 as (check this satisfies (387))

$$\begin{aligned} x^0 &= a\cosh\chi, \qquad x^1 = a\sinh\chi\sin\theta\cos\phi, \\ x^2 &= a\sinh\chi\sin\theta\sin\phi, \qquad x^3 = a\sin\chi\cos\theta \end{aligned} \tag{388}$$

where $\chi > 0$, $0 < \theta < \pi$ and $0 < \phi < 2\pi$. Pulling back the Minkowski metric gives

$$ds^2 = a^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{389}$$

(χ, θ, ϕ) are analogous to spherical polar coordinates in Euclidean space. One can introduce new charts to cover places where $\chi = 0$, $\theta = 0, \pi$ or $\phi = 0, 2\pi$.

3. On S^3 set $r = \sin\chi \in (0,1)$ and on H^3 set $r = \sinh\chi \in (0,\infty)$. The metric becomes

$$ds^2 = a^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right] \tag{390}$$

where $k = +1$ for S^3, $k = -1$ for H^3 and $k = 0$ gives 3d Euclidean space.

Theorem. If (M_1, g_1) and (M_2, g_2) are n-dimensional with metrics of the same signature, and have constant curvature with the same value for K then they are *locally isometric*: for any $p \in M_1$ there exists a neighbourhood $\mathcal{O}$ of p and an isometry $\phi : \mathcal{O} \to \mathcal{O}'$ for some $\mathcal{O}' \subset M_2$.

Corollary. A Lorentzian manifold with vanishing Riemann tensor is locally isometric to Minkowski spacetime. A constant curvature Riemannian manifold is locally isometric to

S^n with round metric of radius $a = \sqrt{1/K}$	if $K > 0$
$\mathbb{R}^n$ with Euclidean metric	if $K = 0$
$\mathbb{R}^n$ with hyperbolic metric of radius $a = \sqrt{-1/K}$	if $K < 0$

Example. Consider the 2d manifold $\mathbb{R} \times S^1$ (an infinite cylinder) with metric $ds^2 = dx^2 + d\theta^2$ where x is a coordinate on $\mathbb{R}$ and θ is a coordinate on S^1. This metric is flat and hence constant curvature. The manifold is locally, but not globally, isometric to Euclidean space.

Remark. Consider a 4d Lorentzian manifold of constant positive curvature. Contracting (382) gives

$$R_{ab} = 3K g_{ab} \qquad R = 12K \qquad \Rightarrow \qquad G_{ab} = -3K g_{ab} \tag{391}$$

Hence such a manifold satisfies the vacuum Einstein equation with non-vanishing cosmological constant $\Lambda = 3K$.

44 De Sitter Spacetime

Consider $\mathbb{R}^5$ with Minkowski metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \ldots (dx^4)^2 \tag{392}$$

Now consider the surface defined by

$$-(x^0)^2 + (x^1)^2 + \dots (x^4)^2 = L^2 \qquad (393)$$

where $L > 0$. This is a 1-sheeted hyperboloid:

The pull-back of the Minkowski metric to this surface is called the *de Sitter metric of radius L*. The surface with this metric is called *de Sitter spacetime.*

Definition. A manifold with metric is *homogeneous* if, for any $p, q \in M$ there exists an isometry ϕ such that $\phi(p) = q$.

Remark. A homogeneous spacetime is the same everywhere. There is no way of distinguishing one point from any other. S^n with round metric, hyperbolic space, Euclidean space and Minkowski spacetime are all homogeneous.

Proposition. De Sitter spacetime is homogeneous.
Proof. Consider a 5d Lorentz transformation $\Lambda : \mathbb{R}^5 \to \mathbb{R}^5$, $x^\alpha \mapsto \Lambda^\alpha{}_\beta x^\beta$. This is an isometry of the 5d Minkowski metric η: $\Lambda_*(\eta) = \eta$. Let M denote the hyperboloid and ι denote the inclusion map which sends a point on M to the corresponding point in $\mathbb{R}^5$. Since Λ preserves $\eta_{\alpha\beta} dx^\alpha dx^\beta$, is also preserves $\eta_{\alpha\beta} x^\alpha x^\beta$ and hence the equation defining M is invariant under Λ. Therefore Λ maps points on M to points on M in a smooth invertible way, hence Λ defines a diffeomorphism $\tilde{\Lambda} : M \to M$. By definition we have $\iota \circ \tilde{\Lambda} = \Lambda \circ \iota$. The metric on M is the pull-back of η: $g = \iota^*(\eta)$. Hence we have

$$\tilde{\Lambda}^*(g) = \tilde{\Lambda}^*(\iota^*(\eta)) = (\iota \circ \tilde{\Lambda})^*(\eta) = (\Lambda \circ \iota)^*(\eta) = \iota^*(\Lambda^*(\eta)) = \iota_*(\eta) = g \quad (394)$$

Hence $\tilde{\Lambda}$ is an isometry of g. M can be viewed as the set of spacelike vectors of norm L in $\mathbb{R}^5$. Any two such vectors are related by a Lorentz transformation Λ. Hence any two points of M are related by an isometry $\tilde{\Lambda}$.

Remark. It can be shown that these isometries are the *only* (continuous) isometries of de Sitter spacetime. Hence the full isometry group is the same as the group of Lorentz transformations of 5d Minkowski spacetime. This group is denoted $SO(4,1)$. 5d Lorentz transformations correspond to a family of Killing vector fields specified by a constant antisymmetric matrix (see examples sheet 3 for the 4d case) and hence have 10 independent parameters. It follows that $SO(4,1)$ has 10 independent parameters, so de Sitter spacetime has 10 linearly independent Killing vector fields, just like 4d Minkowski spacetime: it is *maximally symmetric*. (A n dimensional manifold with metric is maximally symmetric if there are $n(n+1)/2$ linearly independent Killing vector fields.)

We can introduce coordinates (t, χ, θ, ϕ) on the hyperboloid as follows:

$$\begin{aligned} x^0 &= L\sinh(t/L), \qquad x^1 = L\cosh(t/L)\sin\chi\sin\theta\cos\phi, \\ x^2 &= L\cosh(t/L)\sin\chi\sin\theta\sin\phi, \qquad x^3 = L\cosh(t/L)\sin\chi\cos\theta, \\ x^4 &= L\cosh(t/L)\cos\chi \end{aligned} \tag{395}$$

Using these coordinates, the de Sitter metric is (exercise)

$$\begin{aligned} ds^2 &= -dt^2 + L^2\cosh^2(t/L)\left(d\chi^2 + \sin^2\chi d\Omega^2\right) \\ &= -dt^2 + L^2\cosh^2(t/L)\, d\Omega_3^2 \end{aligned} \tag{396}$$

where $d\Omega_3^2$ is the round metric on S^3 of unit radius ($a = 1$). As a manifold, our surface is $\mathbb{R} \times S^3$ where t is a coordinate on $\mathbb{R}$. The t coordinate is globally defined and (χ, θ, ϕ) are the S^3 coordinates we defined earlier. These coordinates are called *global coordinates* for de Sitter spacetime. A surface of constant t is a S^3 of radius $L\cosh t/L$. The radius of the S^3 increases with t if $t > 0$. In this sense, de Sitter spacetime describes an expanding universe (contracting if $t < 0$). However, this is an artifact of our choice of coordinates: all points in de Sitter spacetime are equivalent.

The de Sitter metric is a metric of constant curvature with $K = 1/L^2$ and hence it solves the Einstein equation with

$$\Lambda = \frac{3}{L^2} \tag{397}$$

It follows from the theorem above that any other 4d spacetime with constant positive curvature must be locally isometric to de Sitter spacetime. We shall see that current data suggests that our Universe will approach de Sitter spacetime in the far future.

Writing out the geodesic deviation equation in a parallelly transported frame (314), and using the constant curvature condition gives

$$\frac{d^2 S_0}{d\tau^2} = 0, \qquad \frac{d^2 S_i}{d\tau^2} = \frac{1}{L^2} S_i. \tag{398}$$

If an observer sets up test particles so that they are initially at rest relative to him ($dS_\alpha/d\tau = 0$ at $\tau = 0$) then their proper distance from him increases exponentially: $S_i = S_i(0)\cosh(\tau/L)$.

Exercise. Show that there is a family of timelike geodesics in de Sitter spacetime given by

$$t = \tau, \qquad \chi, \theta, \phi = \text{constant} \tag{399}$$

where τ is proper time. Show that there is a family of null geodesics with

$$\chi = \text{constant} \pm 2\tan^{-1} e^{t/L}, \qquad \theta, \phi = \text{constant} \tag{400}$$

Remark. Any timelike or null geodesic can be mapped to one of these by acting with an isometry. One way to prove this is to argue that one can find an isometry that maps a point on a geodesic and the tangent vector to the geodesic at that point to the corresponding quantities at, say, $\tau = 0$ on one of the above geodesics. From uniqueness of the solution to the geodesic equation and the fact that isometries map geodesics to geodesics, it follows that the full geodesic must coincide with one of the above.

The easiest way to understand the global structure of de Sitter space is to introduce the notion of a Penrose diagram. Consider two Lorentzian metrics on a manifold M: the physical spacetime metric g_{ab} and an unphysical metric $\hat{g}_{ab}$ related as follows

$$\hat{g}_{ab} = \Omega^2 g_{ab} \tag{401}$$

where Ω is a positive function on M. The metric $\hat{g}_{ab}$ is said to be related to g_{ab} by a *conformal transformation.* Note that the lightcones defined with respect to $\hat{g}_{ab}$ are the same as those defined with respect to g_{ab} i.e., they agree on which vectors are timelike, null and spacelike. Hence there exists a timelike or null curve connecting two points in (M, g) iff there exists such a curve between then same points in $(M, \hat{g})$. We say that g_{ab} and $\hat{g}_{ab}$ have the same *causal structure.*

The idea is to choose the function Ω to make the causal structure of $(M, \hat{g})$ obvious. Let's use de Sitter spacetime as an example. Start by writing the de Sitter metric as

$$ds^2 = L^2 \cosh^2(t/L) \left[-\frac{dt^2}{L^2 \cosh^2(t/L)} + d\Omega_3^2 \right] \tag{402}$$

Now define a new "conformal" time coordinate η

$$\eta = 2 \tan^{-1} e^{t/L} \qquad \Rightarrow \qquad d\eta = \frac{dt}{L \cosh(t/L)} \tag{403}$$

hence $\eta \in (0, \pi)$. Now $L^2 \cosh^2(t/L) = 1/\sin^2 \eta$ and hence

$$ds^2 = \frac{1}{\sin^2 \eta} \left(-d\eta^2 + d\Omega_3^2 \right) \tag{404}$$

Now choosing

$$\Omega = \sin \eta \tag{405}$$

makes the unphysical metric very simple:

$$\hat{g} = -d\eta^2 + d\Omega_3^2 \tag{406}$$

In this metric, there is nothing to stop us from extending η to $\pm\infty$. The resulting spacetime is called the *Einstein static universe*. The spacetime manifold is $\mathbb{R} \times S^3$ in which the time direction is $\mathbb{R}$. Note that the metric is independent of the time coordinate η, i.e., $\partial/\partial\eta$ is a Killing vector field. The above analysis shows that de Sitter spacetime is conformal to the portion $0 < \eta < \pi$ of the Einstein static universe. If we suppress the S^2 directions parameterized by θ, ϕ we can draw this as:

In this diagram, we're depicting S^3 to S^1 owing to our inability to sketch 4d manifolds.

Note that $\Omega = 0$ at $\eta = 0, \pi$, i.e., at $t = \pm\infty$. One can check that $t = \pm\infty$ really is ”at infinity” in de Sitter spacetime, in the sense that any timelike or null geodesic reaches $t = \pm\infty$ in the limit of infinite affine parameter. Hence the surfaces $\eta = 0, \pi$ in the Einstein static universe correspond to past and future infinity in de Sitter spacetime. We denote these surfaces as $\mathcal{I}^-$ and $\mathcal{I}^+$. Note that they are 3-spheres. By choosing Ω appropriately we have ”conformally compactified” the spacetime, mapping points at infinity in the original spacetime to points in the interior of the new spacetime.
If we flatten the above diagram and just plot the (η, χ) plane then we obtain the *Penrose diagram* for de Sitter spacetime:

Each point on this diagram represents a S^2 of area $4\pi \sin^2 \chi / \sin^2 \eta$. Note that null curves with constant (θ, ϕ) have $d\eta^2 = d\chi^2$, i.e., $\chi = \text{const} \pm \eta$ (these are the geodesics (400)) and hence are straight lines at $45°$ to the horizontal in this diagram. These define ”light cones” at each point of the diagram. All other timelike or null curves must have tangent vectors lying inside the light cone at each point. (Null geodesics which have non-trivial velocity in the θ or ϕ directions would appear timelike if we plotted their trajectories in the (η, χ) plane.) Timelike and null curves start at $\mathcal{I}^-$ and end at $\mathcal{I}^+$.
Consider an observer, Alice, in de Sitter spacetime. By acting with an isometry, we can assume that her worldline is the timelike geodesic with $\chi = 0$, $t = \tau$, i.e., the left edge of the Penrose diagram. Consider the point p on the wordline of A. Because nothing can travel faster than light, the only part of the spacetime from which A could have received a signal by the time she reaches p is the shaded region:

The boundary of this region is the surface $\eta = \eta_p - \chi$. This region is called the *past light cone of* p. It is the same for all observers at p. To see this, note that if A can receive a signal from a point q before, or at, p then she can pass it on to any other observer at p. In this way, a signal has travelled from q to the other observer.
Consider the worldine of another observer, Bob, shown. It lies outside the past light cone of p. Hence there is no way that Alice could determine whether or not Bob exists by the time she reaches p. Constrast this with the situation in Minkowski spacetime where, if Bob sends a light signal at a sufficiently early time, then it will reach Alice by the time she reaches a given event p. Hence the far past of Bob's worldline is always visible to Alice in Minkowski spacetime. Not so in de Sitter spacetime.
Now let p approach $\mathcal{I}^+$:

The shaded region is the region from which A can receive a signal at any point along her worldine. The boundary of this region is called the *cosmological horizon*. Points outside this horizon are invisible to A even if she waits for an arbitrarily long time. The cosmological horizon depends (only) on p. However, we are usually most interested in the particular horizon defined using our own worldline.

Consider Bob again. Even if A waits for an arbitrarily long time, she cannot see points beyond q on Bob's worldline because they are outside her cosmological horizon. It takes infinitely long for her to see Bob reach the event q. This is very similar to the infinite redshift effect that we saw when we discussed black holes.
There is another coordinate system that is frequently used to describe de

Sitter spacetime. It is defined by

$$\hat{t} = L \log\left(\frac{x^0 + x^4}{L}\right), \qquad \hat{x}^i = \frac{Lx^i}{x^0 + x^4} \tag{407}$$

where $i = 1, 2, 3$. These coordinates cover only the half of de Sitter spacetime with $x^0 + x^4 > 0$. In these coordinates, the de Sitter metric is (examples sheet 4)

$$ds^2 = -d\hat{t}^2 + e^{2t/L}\delta_{ij} d\hat{x}^i d\hat{x}^j \tag{408}$$

Surfaces of constant $\hat{t}$ are flat: this is called the *flat slicing* of de Sitter spacetime. Note that $x^0 + x^4$ is a function of t and χ in the global coordinate discussed above. Hence $\hat{t}$ is a function of t and χ or, equivalently, a function of η and χ.

Exercise. Show that $\hat{t} = -\infty$ (i.e. $x^0 + x^4 = 0$) corresponds to $\eta = \chi$. Plotting surfaces of constant $\hat{t}$ on the Penrose diagram of de Sitter spacetime gives

45 FLRW cosmology

The Copernican principle asserts that we do not occupy a privileged position in the Universe: the Universe viewed from one point should "look the same" as when viewed from another point. Clearly these features are not true on small scales but, on the largest length scales (above 10^8 light years), observations of the distribution of galaxies provides strong evidence that the Copernican principle is correct. Mathematically, we interpret the Copernican principle as the statement that spacetime is *spatially* homogeneous.

Definition. A spacetime (M, g) is *spatially homogeneous* it there exists a group of isometries whose orbits are 3d spacelike surfaces.

Recall that the orbit of a point p is the set of points obtained by acting with the isometry group on p. We say that a surface is spacelike if any vector tangent to the surface is spacelike. Pulling back the spacetime metric gives a homogeneous metric on each of these surfaces. Hence through each point of a spatially homogeneous spacetime there exists a 3d surface with a homogeneous metric. This is to be regarded as "space" at a given "instant of time".
Our Universe has another important property: it looks the same in all directions, i.e., it is *isotropic*. The best evidence for this comes from the observed uniformity of the cosmic microwave background radiation. Note that only certain observers can see the universe as isotropic. If an observer at p has a 4-velocity which has a non-trivial component along the surface of spatial homogeneity through p then this picks out a spatial direction within this surface. But a preferred spatial direction is incompatible with isotropy. Hence only observers with 4-velocity normal to the surfaces of spatial homogeneity can perceive the Universe to be isotropic. This gives us a preferred class of observers, called *comoving observers*.
Since comoving observers exist everywhere in the spacetime, their 4-velocity determines a vector field u^a. This is just the unit normal to the surfaces of homogeneity. It can be shown that one can introduce coordinates (t, x^i) such that the spacetime metric takes the form

$$ds^2 = -dt^2 + a(t)^2 h_{ij}(x) dx^i dx^j \tag{409}$$

where $u^a = (\partial/\partial t)^a$ and the surfaces of homogeneity are given by $t =$ constant, with metric $a(t)^2 h_{ij}(x)$. Isotropy can be defined as the statement that given two unit (spacelike) vectors v^a and w^a that are tangent to such a surface at some point p, there exists an isometry ϕ such that $\phi_*(v) = w$. It can be shown that spatial homogeneity and isotropy imply that these surfaces must be spaces of constant curvature and hence locally we have

$$ds^2 = -dt^2 + a(t)^2 d\Sigma_k^2 \tag{410}$$

where

$$d\Sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{411}$$

where $k = 1, 0, -1$ corresponds to S^3, Euclidean space or H^3. The function $a(t)$ is called the *scale factor*. A spacetime with metric (410) is called Friedmann-Lemaitre-Robertson-Walker (FLRW) universe (or sometimes a Friedmann universe, or a FRW universe, or a RW universe, ...). It is a good approximation to the geometry of our Universe on the largest scales. The universe is said to be *closed* if $k = 1$, *flat* if $k = 0$ and *open* if $k = -1$.

Exercise. Comoving observers have worldines with constant x^i and $t = \tau$ (proper time). Show that such curves are geodesics.

Consider two comoving particles ("galaxies"). At any instant of time t, the proper distance between then is $d = a(t)R$ where R is the distance between then calculated using the time-independent metric h_{ij}. For example, if $k = 0$ we can use Cartesian coordinates so that $h_{ij} = \delta_{ij}$ and then $R = \sqrt{\Delta x^i \Delta x^i}$ where Δx^i is their coordinate separation. It follows that the "relative velocity" of the two particles is

$$v \equiv \dot{d} = \dot{a}R = Hd \tag{412}$$

where a dot denotes a t-derivative and

$$H = \frac{\dot{a}}{a}. \tag{413}$$

Equation (412) states that at any time, we should observe the velocity of a galaxy to be proportional to its distance. This is known as *Hubble's law*. The current value of H, denotes H_0, is the *Hubble constant*. Hubble found that galaxies are moving apart, so $a(t)$ is increasing with time. Note that $v > 1$ for a very distant galaxy. This does not contradict the assertion that "nothing can move faster than light", which refers to the relative velocity of two particles at the same point.
To determine $a(t)$ we need to solve Einstein's equation. What energy-momentum tensor should we put on the RHS? If we are interested only in the largest length scales then we can approximate galaxies as the particles of a fluid, with some energy density ρ. Galaxies interact only gravitationally hence we can treat the fluid as pressureless. Furthermore, since the distribution of galaxies is observed to be homogeneous and isotropic, they must be comoving, i.e., have 4-velocity $u^a = (\partial/\partial t)^a$, and ρ cannot depend on x^i. A pressureless perfect fluid is usually called *dust* although cosmologists also call it *matter*. Recall that the fluid equations imply that the 4-velocity u^a of such a fluid must be geodesic, as is the case for our assumed comoving fluid.

The Universe also contains electromagnetic radiation, which forms the cosmic microwave background radiation. This is thermal, with a temperature of about $2.7K$. Blackbody radiation can be described as a perfect fluid with $p = \rho/3$. Once again, homogeneity and isotropy implies that $u^a = (\partial/\partial t)^a$ for this fluid.
We also have the cosmological constant. As discussed previously, this can be regarded as a perfect fluid with $p = -\rho = \Lambda/(8\pi G)$. Hence all of the contributions to the energy-momentum tensor can be described by a perfect fluid, but with different *equations of state* (an equation of state is a rule specifying p as a function of ρ). However, each is linear and therefore we can summarize as

$$p = w\rho \tag{414}$$

where $w = 0$ for dust/matter, $w = 1/3$ for radiation and $w = -1$ for a cosmological constant.
Conservation of the energy-momentum tensor $\nabla_a T^{ab} = 0$ reduces to the equation (exercise)

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{415}$$

which can also be written

$$\frac{d}{dt}\left(a^3\rho\right) = -p\frac{d}{dt}\left(a^3\right) \tag{416}$$

Note the the LHS is the rate of increase of energy in a fixed comoving volume and the RHS is the rate of working against the pressure of the fluid as it expands ($dE = -pdV$).
The separate fluids (dust, radiation, Λ) have energy momentum tensors that are separately conserved (this arises from the assumption that interactions between these fluids are negligible). Hence each must obey (415). Substituting in $p = w\rho$ for constant w we can integrate to obtain

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^{3(1+w)} \tag{417}$$

In cosmology, a subscript 0 refers to the value of the corresponding quantity at the present time. As the Universe expands, the energy density of the different fluids dilutes at different rates: dust as a^{-3}, radiation as a^{-4} and a cosmological constant does not dilute at all. It follows that if the Universe expands indefinitely then ultimately it will be dominated by Λ.

With a perfect fluid on the RHS of the Einstein equation, it can be shown to reduce to the pair of equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{418}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \tag{419}$$

Equation (418) is called the *Friedmann equation.* Equation (419) can be derived from the Friedmann equation and equation (415).
What is the value of k in our Universe? The spatial geometry is a space of constant curvature with $K = k/a^2$. Observations can measure K, from which one finds that the term k/a^2 in Friedmann's equation is negligible compared to the other term on the RHS of the equation at the present time. For matter, or radiation, the other term grows faster that k/a^2 as $a \to 0$, hence it follows that the curvature term was even more unimportant early on in the Universe. Hence it is a good approximation to set $k = 0$.
Since the different types of fluid dilute at different rates, they will be important at different epochs of the Universe. Radiation will dominate at early times and a cosmological constant will dominate at late times, with matter domination possible at intermediate times. Therefore let's consider the case in which a single fluid dominates in the Friedmann equation. We can use (417) to obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)} - \frac{k}{a^2} \tag{420}$$

This ODE can be integrated to determine $a(t)$ for different equation of state parameters w. The simplest case is $w = -1$, a cosmological constant:

Exercise. Solve this equation for a (positive) cosmological constant ($w = -1$). Show that the resulting Universe is de Sitter spacetime for $k = 1, 0$. (The same is true for $k = -1$ but we have not discussed the "open slicing" of de Sitter.) A constant of integration can be eliminated by a shift in the time coordinate: $t \to t + \text{constant}$,

Another simple case is a flat universe ($k = 0$) for which (assuming $\dot{a} > 0$ and $w > -1$ and shifting t to eliminate a constant of integration)

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{421}$$

Note that $a(t)$ is a monotonically increasing function of t with $a(t) \to 0$ as $t \to 0+$. Hence if we follow the evolution of this universe backwards, the proper distance between particles decreases, and tends to zero as $t \to 0+$. The energy density ρ diverges at $t = 0$. Since this is a scalar, it diverges in all coordinate charts. Hence the spacetime cannot be extended smoothly beyond $t = 0$. It can be shown that the Ricci scalar also diverges at $t = 0$ hence $t = 0$ is a curvature singularity. This is called the Big Bang.
Many people used to think that the Big Bang singularity was an effect of assuming exact homogeneity and isotropy. However, the *singularity theorems* of Hawking and Penrose proved that a past singularity must occur for a universe that is close to being homogeneous and isotropic today, assuming that the energy-momentum tensor of matter satisfies a "reasonable" condition called the strong energy condition, which is equivalent to $\rho + p \geq 0$, $\rho + 3p \geq 0$ for a perfect fluid. Hence there must have been a singularity if matter or radiation dominated early in the universe. (Theories of "inflation" assume that the strong energy condition was violated in the very early universe and thereby evade the singularity theorems.)

46 Causal structure of FLRW universe

It is convenient to introduce a *conformal time* coordinate η defined by

$$\eta = \int_0^t \frac{dt'}{a(t')} \qquad \Rightarrow \qquad d\eta = \frac{dt}{a(t)} \tag{422}$$

hence the FLRW metric becomes

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + d\Sigma_k^2\right) \tag{423}$$

where $a(\eta) = a(t(\eta))$. An immediate consequence is that the FLRW metric has the same causal structure as the *time-independent* unphysical metric (choose $\Omega = 1/a$ in (401))

$$\hat{g} = -d\eta^2 + d\Sigma_k^2 \tag{424}$$

From examples sheet 4, g and $\hat{g}$ have the same null geodesics. Consider two photons, the first emitted by a galaxy at time η_e and observed by us at time η_o. The second is emitted at time $\eta_e + \Delta\eta$. Since the unphysical geometry is time independent, its path must be the same as that of the first photon translated in time by $\Delta\eta$. Hence it is observed by us at time $\eta_o + \Delta\eta$.

Since the galaxies are comoving, it follows that the photons are emitted from points with the same values for the coordinates x^i. Hence the proper time between the emission of the two photons is given by $(\Delta\tau_e)^2 = a(\eta_e)^2(\Delta\eta)^2$, where we assume that $\Delta\eta$ is small compared to the scale of time variation of a. So $\Delta\tau_e = a_e\Delta\eta$ where $a_e = a(\eta_e)$. Similarly, the proper time between the reception of the the photons is $\Delta\tau_o = a_o\Delta\eta$ where $a_o = a(\eta_o)$. Hence

$$\frac{\Delta\tau_0}{\Delta\tau_e} = \frac{a_o}{a_e} > 1 \tag{425}$$

The proper time interval between the observed photons is greater than that between the emitted photons. If instead of photons, we apply the argument to successive wavecrests of a light wave, we see that the received wave has a greater period than the emitted wave so the light undergoes a *redshift*. By measuring the redshift of light from a distant galaxy, we can deduce how much smaller the Universe was when that light was emitted. The largest observed redshifts for galaxies are $a_0/a_e \sim 9$.

In terms of η, (420) becomes

$$\left(\frac{da}{d\eta}\right)^2 + ka^2 = C^2 a^{1-3w} \tag{426}$$

where a prime denotes an η-derivative and $C > 0$ is defined by $C^2 = 8\pi\rho_0 a_0^{3(1+w)}/3$. The simplest case is radiation ($w = 1/3$) for which

$$a(\eta) = \begin{cases} C\sin\eta & \text{if } k = 1 \\ C\eta & \text{if } k = 0 \\ C\sinh\eta & \text{if } k = -1 \end{cases} \tag{427}$$

where we have assumed that $a' > 0$ and shifted η so that $a = 0$ at $\eta = 0$. Sketching the different cases gives:

All three cases start with a Big Bang singularity, in accord with the singularity theorems. The closed universe stops expanding at $\eta = \pi/2$ and recollapses into a "Big Crunch" singularity at $\eta = \pi$. The flat and open universes expand forever, although the rate of expansion is decreasing (equation (419) implies that $\ddot{a} < 0$).
In the case of a matter dominated universe ($w = 0$), the solution is

$$a(\eta) = \begin{cases} (C^2/2)(1 - \cos\eta) & \text{if } k = 1 \\ (C^2/4)\eta^2 & \text{if } k = 0 \\ (C^2/2)(\cosh\eta - 1) & \text{if } k = -1 \end{cases} \tag{428}$$

Once again, each case starts with a Big Bang. The closed case recollapses into a Big Crunch at $\eta = 2\pi$ whereas the flat and open cases expand forever, with deceleration ($\ddot{a} < 0$).
In the case of a closed universe, the unphysical metric (424) is the portion $0 < \eta < \pi$ (radiation dominated) or $0 < \eta < 2\pi$ (matter dominated) of the Einstein static universe, where the boundaries are now curvature singularities. Hence we can draw Penrose diagrams just as we did for de Sitter spacetime:

Here χ is the coordinate on S^3 we discussed previously. Consider a comoving observer, who we can assume to be at $\chi = 0$ (any comoving observer has constant χ and all points on S^3 are equivalent). The cosmological horizon for such an observer is shown.
In the $k = 0$ case, the unphysical metric is Minkowski spacetime

$$\hat{g} = -d\eta^2 + dx^2 + dy^2 + dz^2 \tag{429}$$

where we have traded spherical polar coordinates on $d\Sigma_0^2$ for Cartesian coordinates. This is a considerable simplification, but it is not a conformal compactification because the ranges of all of the coordinates are still infinite. To draw the Penrose diagram for this case we would have to understand how to draw the Penrose diagram for Minkowski spacetime (see the black holes course). Nevertheless, we can use the unphysical metric to understand the causal structure of a $k = 0$ universe.
In the case of a single type of fluid, we can use equation (421) to determine, for $w > -1/3$,

$$a(\eta) = a_0 \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1+3w}} \tag{430}$$

The Big Bang corresponds to $\eta = 0$ and there is no upper limit on η. Therefore the causal structure of the spacetime is the same as that of Minkowski spacetime but with the restriction $\eta > 0$. If an observer waits for long enough, she can receive a signal from any point in the spacetime so there is no cosmological horizon. However, if we consider a flat universe that is radiation or matter dominated early on (and hence a Big Bang at $\eta = 0$) but Λ dominated at late times, i.e., $a(t) \propto e^{t/L}$ for large t, then from (422) we have $\eta \to \eta_\infty$ (a constant) as $t \to \infty$. Hence the physical spacetime has $0 < \eta < \eta_\infty$, which implies that there *is* a cosmological horizon (as would be expected, since the universe is approaching de Sitter spacetime):

Now let's not worry about what happens in the far future and consider the history of our Universe, which we assume to be matter or radiation dominated at early times. Comoving particles (i.e. galaxies) follow lines of constant x, y, z. Consider the point p shown:

Only comoving particles whose worldines intersect the past light cone of p can send a signal to an observer at p. The boundary of the region containing such worldines is called the *particle horizon at p*. This is what cosmologists usually mean when they refer to "the horizon" (with p our present spacetime location). The only galaxies visible at p are those inside the particle horizon at p.

Exercise. Show that a closed radiation dominated universe has a particle horizon for any p. Show that a closed matter dominated universe has a particle horizon at p only when p is in the "expanding phase" ($\eta < \pi$) of the evolution. (If you consider points p that are not at $\chi = 0$ or $\chi = \pi$ then it helps to visualize the Einstein static universe as a cylinder instead of using the Penrose diagram.)

The existence of particle horizons leads to a puzzle. The cosmic microwave background radiation (CMB) is left over from a time a few hundred thousand years after then Big Bang. Its temperature is very uniform: $2.7K$ with deviations of order 10^{-5}. At earlier times, the Universe contained an ionized plasma of electrons and protons, which interacted with photons. As the Universe expanded, it cooled, and the electrons and protons combined into neutral (hydrogen) atoms, a process misleadingly called *recombination.* After recombination, the photons stopped interacting significantly with matter, and gradually cooled. These are the photons that form the CMB that we observe today. Hence observations of the CMB are looking back to the time of recombination. Consider two opposite directions on the sky and let q,

r be the points (at the time of recombination) at which photons in those directions were emitted, to reach us at event p:

The photons were emitted at a time sufficiently close to the Big Bang that there is no point that lies inside the particle horizons of both p and q. Hence no point can send a signal to both p and q. So how do photons at p and q "know" that they should be at almost exactly the same temperature? This is the *horizon problem*. A proposed solution to this problem is that the very early universe underwent a period of *inflation* (before it became radiation dominated), which is a de Sitter like phase with $a(t) \propto e^{t/L}$. This has the effect of greatly decreasing the lower bound on η and thereby increasing the size of particle horizons.