Question 14

The probability of throwing a 6 with a biased die is p. It is known that p is equal to one or other of the numbers A and B where $0 < A < B < 1$. Accordingly the following statistical test of the hypothesis $H_0 : p = B$ against the alternative hypothesis $H_1 : p = A$ is performed.

The die is thrown repeatedly until a 6 is obtained. Then if X is the total number of throws, H_0 is accepted if $X \leq M$, where M is a given positive integer; otherwise H_1 is accepted. Let α be the probability that H_1 is accepted if H_0 is true, and let β be the probability that H_0 is accepted if H_1 is true.

Show that $\beta = 1 - \alpha^K$, where K is independent of M and is to be determined in terms of A and B.

Using conditional probability, we have

$$\alpha = \mathrm{P}\left(H_1 \text{ accepted} | H_0 \text{ is true}\right) = \mathrm{P}\left(X > M | p = B\right)$$
$$= \frac{\mathrm{P}\left(X > M \text{ and } p = B\right)}{\mathrm{P}(p = B)}$$

and similarly

$$\beta = \mathrm{P}\left(H_0 \text{ accepted} | H_1 \text{ is true}\right) = \mathrm{P}\left(X \leq M | p = A\right)$$
$$= \frac{\mathrm{P}\left(X \leq M \text{ and } p = A\right)}{\mathrm{P}(p = A)}$$

Breaking up the probability for β and summing the individual conditional probabilities, we have

$$\beta = \sum_{i=1}^{M} \mathrm{P}\left(X = i | p = A\right) = \sum_{i=1}^{M} \frac{\mathrm{P}\left(X = i \text{ and } p = A\right)}{\mathrm{P}(p = A)}$$
$$= \frac{A\left(A + A\left(1 - A\right) + A\left(1 - A\right)^2 + \cdots + A\left(1 - A\right)^{M-1}\right)}{A}$$
$$= \sum_{i=1}^{M} A\left(1 - A\right)^{i-1} = \frac{A\left\{1 - \left(1 - A\right)^M\right\}}{1 - (1 - A)}$$
$$= 1 - \left(1 - A\right)^M .$$

Similarly for α,

$$\alpha = 1 - \sum_{i=1}^{M} \mathrm{P}\left(X = i | p = B\right) = 1 - \left\{1 - \left(1 - B\right)^M\right\} = \left(1 - B\right)^M .$$

Since $\beta = 1 - \left(1 - A\right)^M$, hence let $\alpha^K = (1 - A)^M$ gives

$$\left(1 - A\right)^M = \left(1 - B\right)^{MK}$$

taking logs on both sides and we've found K in terms of A and B.

$$K = \frac{\ln(1-A)}{\ln(1-B)}.$$

Sketch the graph of β against α.

If we subtract the inequality $0 < A < B < 1$ from 1, we have

$$1 > 1 - A > 1 - B > 0$$

hence we establish that $K > 1$.

Differentiating wrt. α gives

$$\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = -K\alpha^{K-1}$$

so when $\alpha = 0$, $\dfrac{\mathrm{d}\beta}{\mathrm{d}\alpha} = 0$ and negative gradient everywhere else.