P6 summary

Series

Maclaurin's expansion states:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^r}{r!}f^r(0) + ...$$

Taylor's series for $f(x + a)$ in ascending powers of x is:

$$f(x + a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + ... + \frac{x^r}{r!}f^r(a) + ...$$

Taylor's series for $f(x)$ in ascending powers of $(x - a)$ is:

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + ... + \frac{(x - r)^r}{r!}f^r(a)$$

Forming a Taylor series solution to a differential equation merely involves finding the values of $y_a, (\frac{dy}{dx})_a, (\frac{d^2y}{dx^2})_a$...and so on, then forming the series in ascending powers of $(x - a)$ using the above results. Often, $a = 0$ and we can simply use Maclaurin's expansion.

Sometimes we can approximate an integral by expanding the integrand as a polynomial using Taylor's series.

Complex numbers

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta} where - \pi < \theta \leq \pi$$

$$\cos iz = \cosh z$$

$$\sin iz = i\sinh z$$

$$\cosh iz = \cos z$$

$$\sinh iz = i\sin z$$

When we multiply two complex numbers we multiply moduli and add the arguments.
When we divide two complex numbers we divide the modulus of the numerator by the modulus of the denominator and subtract the argument of the denominator from the argument of the numerator.

De Moivre's theorem states that if:

$$z = r(\cos\theta + i\sin\theta)$$

then:

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

When using De Moivre's theorem to find roots of $z^n = a + ib$ remember to express $a + ib$ in modulus argument form and add $2k\pi$ to the argument before applying the power $\frac{1}{n}$ to both sides.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$
$$|z_1 + z_2| \geq |z_1| - |z_2|$$

Matrix algebra

A linear transformation T is such that $T(a_1v_1 + a_2v_2) = a_1T(v_1) + a_2T(v_2)$
If a linear transformation T is one-one then an inverse transformation T^{-1} exists.

If A and B are matrices, then in general $AB \neq BA$

The identity matrix I consists of a matrix of zeros except that all cells in the leading diagonal are occupied by 1.

If AB=I, then $\mathrm{B}=\mathrm{A}^{-1}$ and $\mathrm{A}=\mathrm{B}^{-1}$.

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \frac{1}{ps - qr} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$$

Note the determinant is $ps - qr$

To find the tranpose of a matrix simply swap rows and columns.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

To find the inverse of a 3x3 matrix: (i) Form the matrix of minors (replacing each element with the determinant of the 2x2 matrix formed when its rows and columns are crossed out) (ii) Form the matrix of cofactors (applying the law of alternating signs) (iii) Transpose the matrix of cofactors (iv) Divide the matrix of cofactors by the determinant of the original matrix.

A matrix is singular if its determinant is zero. Only non-singular square matrices have an inverse.

To find a 3x3 matrix which represents a linear transformation T, you find T(i),T(j) and T(k) then make these the first, second and third columns of your matrix.

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(AB)^{T} = B^{T}A^{T}$$

If A is a matrix and v is a non zero vector such that $Av = \lambda v$ then v is called an eigenvector of A and the scalar λ is called the corresponding eigenvalue.

$|A - \lambda I| = 0$ is the characteristic equation of the matrix A. The solutions of this give the eigenvalues of A.

A normalised eigenvector is simply a unit eigenvector.

Two eigenvectors are orthogonal(perpendicular) if their scalar product is 0.

A matrix is orthgonal if the scalar product of any two columns in the matrix is 0. If the matrix is orthogonal then A^T=A^{-1}
A square matrix whose elements are all zero except those on the leading diagonal is called a diagonal matrix.

If A^T =A then A is symmetric.

If A is symmetric and P is an orthagonal matrix whose columns are the normalised, orthogonal eigenvectors of A, then P^TAP is diagonal.
To multiply two matrices you take a row in the first matrix and turn it as if you were to place it next to a column in the second matrix. Multiply adjacent elements, then add up the three numbers. You then take the result of this process and place it at the intersection of the column and row in the new matrix.

Vectors

$$a \times b = |a||b| \sin\theta \widehat{n}$$

$a \times b$ gives a vector perpendicular to both a and b.

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Area Δ ABC $= \frac{1}{2}\left|(\vec{AB}) \times (\vec{AC})\right|$
Area of a parallelipiped is $a.b \times c$, where a, b, c are the vectors connecting one

point with three adjacent points.
The volume of tetrahedron ABCD is $\frac{1}{6}(\vec{AB})(\vec{AC}) \times (\vec{AD})$

The equation of a line through a and r, parallel to the vector b is $(r-a) \times b = 0$

The equation of a plane is given by $r.n = a.n$. R and A are points in the plane and n is a vector perpendicular to the plane.

The vector equation of a plane through a and where b and c are non parallel vectors in the plane can be written as $r = a + \lambda b + \mu c$

The distance from the origin to the a plane is $d = r.\widehat{n}$ where n is a unit vector perpendicular to the plane.

If the angle between a line and a plane is θ and the angle between the line and the perpendicular to the plane is α then $\theta = 90 - \alpha$. We can thus use the scalar product to find the angle between a line and a plane.

The angle between two planes is equal to the angle between the two perpendiculars.

The shortest distance between the lines with equations $r = a+\lambda b$ and $r = c+\mu d$ is given by $\left|\frac{(a-c).b\times d}{|b\times d|}\right|$

Numerical methods

Step by step methods involve setting up a reccurence relation connecting y_{n+1} and previous values of y or x.

$$(\frac{dy}{dx})_0 = \frac{y_1 - y_0}{h}$$

$$(\frac{dy}{dx})_0 = \frac{y_1 - y_{-1}}{2h}$$

$$(\frac{d^2y}{dx^2})_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

If a differential equation involves both $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ then two approximation formulae may need to be used. In this case it is often useful to factorise out y_1 so the reccurence relation is clear.

Occasionally the value of $\frac{dy}{dx}$ and y may be given at a particular value of x, and you are required to approximate y for a differential equation involving $\frac{d^2y}{dx^2}$.

Clearly the approximation for $\frac{d^2y}{dx^2}$ requires two values of y. In this case we use $(\frac{dy}{dx})_0 = \frac{y_1 - y_{-1}}{2h}$ and $(\frac{d^2y}{dx^2})_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2}$. We can use our known values and add the equations to find another value of y so that we can use our reccurence relation

Proof by induction

There are two steps to a proof by mathematical induction:
(i) Prove that if the statement is true for $n = k$ then it is also true for $n = k+1$.
(ii) Prove the statement is true for the first arbtirary value of n such as $n = 1$ or $n = 2$.

There are slight variations however. Sometimes if proving a statement about odd or even numbers, point (i) would be require proof for the case $n = k + 2$ rather than $n = k$.