University of Durham

EXAMINATION PAPER

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LEVEL 3 PHYSICS: THEORETICAL PHYSICS
LEVEL 4 PHYSICS: THEORETICAL PHYSICS 4

SECTION A. QUANTUM MECHANICS
SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM

Time allowed : 3 hours

Examination material provided : None

Answer the compulsory question that heads each of sections A and B. These **two** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **three** of the other questions with **at least one** from each section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.

ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

APPROVED TYPES OF CALCULATOR MAY BE USED.

Information

$e = 1.60 \times 10^{-19}$ C	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
$h = 6.63 \times 10^{-34}$ J s	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Bohr magneton $= 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Nuclear magneton $= 5.05 \times 10^{-27}\ \mathrm{J\,T^{-1}}$	$R = 8.31 \times 10^{3}\ \mathrm{J\,K^{-1}\,kmol^{-1}}$
Avogadro's Constant $= 6.02 \times 10^{26}\ \mathrm{kmol^{-1}}$	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann Constant $= 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$	

SECTION A. QUANTUM MECHANICS

Answer question 1 and **at least one** of questions 2, 3 and 4.

1. (a) Consider an operator $\hat{A}$ on a Hilbert space $\mathcal{H}$. Give the definition of an Hermitian operator.

   Show that, if an operator $\hat{A}$ is Hermitian, the expectation value of $\hat{A}^2$ on a state $|\gamma\rangle$ satisfies $\langle \hat{A}^2 \rangle \geq 0$. [4 marks]

   [Hint: Recall the property of the scalar product: $\langle \phi|\phi\rangle \geq 0$.]

   (b) State the generalised uncertainty relation for two operators $\hat{A}$ and $\hat{B}$.

   Consider a system which is in an eigenstate of $\hat{L}_z$, $|l, m\rangle$. What is the minimal uncertainty in a simultaneous measurement of $\hat{L}_x$ and $\hat{L}_y$? [4 marks]

   (c) Consider a deuterium molecule D_2 whose Hamiltonian is $\hat{H} = \hat{L}^2/(2I)$. Here, $I = 2Ma^2$, with M the mass of the molecule and a the distance between the atoms.

   Write down the commutation relations for the components of the orbital angular momentum, $\hat{L}_i$.

   Which of the sets of observables corresponding to $\{\hat{H}, \hat{L}_z\}$, $\{\hat{H}, \hat{L}_x\}$, $\{\hat{H}, \hat{L}_x, \hat{L}_z\}$, $\{\hat{H}, \hat{L}_x + \hat{L}_y, \hat{L}_z\}$ can be measured simultaneously with infinite precision? [4 marks]

   (d) Consider a deuterium molecule as in (c). At $t = 0$ the system is described by the state

   $$|\psi, t = 0\rangle = \frac{3|1,1\rangle + 4|7,3\rangle + |7,1\rangle}{\sqrt{26}} .$$

   Recall that $|l, m\rangle$ are eigenstates of $\hat{L}^2$ and $\hat{L}_z$ and are normalised.

   At $t = 0$, what are the values of L^2 and L_z a measurement can yield and with what probability?

   At $t = 0$, L_z is measured and the value $3\hbar$ is found. What is the state which describes the system after the measurement? If at a subsequent time t_1 we measure L_x, will the system be in an eigenstate of L_z after the measurement? [4 marks]

   (e) The operator $\hat{b}$ satisfies the following anticommutation relations:

   $$\{\hat{b}, \hat{b}^\dagger\} = \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = \hat{I} \text{ and } \{\hat{b}, \hat{b}\} = 0,\ \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0 .$$

   $\hat{I}$ denotes the identity operator. The operator $\hat{N}$ is defined as $\hat{N} = \hat{b}^\dagger\hat{b}$. Show that $\hat{N}^2 = \hat{N}$ and find the eigenvalues of $\hat{N}$. [4 marks]

   (f) Consider a system with a time-independent Hamiltonian $\hat{H}$. The eigenstates of $\hat{H}$ are $|a_n\rangle$ with eigenvalues E_n. At time $t = 0$ the system is described by the state $|\psi\rangle$.

   State the time-dependent Schrödinger equation and formally solve it.

   If at $t = 0$ the system is an eigenstate $|a_n\rangle$, what is the result of a measurement of the energy at a later time t? [4 marks]

(g) The Born approximation for the scattering amplitude is

$$f^B(\Omega) = -\frac{(2\pi)^2\mu}{\hbar^2}\langle \vec{k}'|V(\vec{r})|\vec{k}\rangle \text{ with } |\vec{k}'| = |\vec{k}|\,.$$

Ω indicates the angles θ and ϕ in spherical coordinates. μ is the reduced mass of the system. For central potentials, $V(r)$, it can be rewritten as $f^B(\Omega) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty dr\ rV(r)\sin(qr)$, where $\vec{q} = \vec{k} - \vec{k}'$.
In the sharp-momentum approximation, find the differential cross section $\frac{d\sigma}{d\Omega}$ for the screened Coulomb potential $V_{sC}(r) = -\frac{e^{-ra}}{r}$, with a constant.

$$[\text{ Hint: } \textstyle\int_0^\infty dr\ e^{-ra}\sin(qr) = \frac{q}{q^2+a^2}.\ ]$$

Using the previous result, compute the differential cross section $\frac{d\sigma}{d\Omega}$ for the Coulomb potential $V_C(r) = -\frac{1}{r}$ and compare it with the exact quantal result:

$$\frac{d\sigma}{d\Omega} = \frac{(2\mu)^2}{(2\hbar k)^4}\frac{1}{\sin^4(\theta/2)}\,.$$

[4 marks]

$$[\text{ Hint: } q = 2k\sin(\theta/2)\ ]$$

2. Consider a non-relativistic particle of mass m in a 3D spherical box. The central potential is given by:

$$V(r) = \begin{cases} 0 & \text{for } r < a \\ \infty & \text{for } r \geq a \end{cases}$$

(a) Write the Hamiltonian for the particle. Does $\hat{H}$ commute with $\hat{L}^2$ and $\hat{L}_z$? Give a short justification for your answer. What does this imply for the eigenstates of the Hamiltonian? [7 marks]

(b) Expressing the eigenfunctions of the Hamiltonian in spherical coordinates as $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$, with $Y_{lm}(\theta,\phi)$ the spherical harmonics, the reduced radial equation for $u_{nl}(r) = rR_{nl}(r)$ is:

$$\left(-\frac{\hbar^2}{2m}\Big(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\Big) + V(r)\right)u_{nl}(r) = E_{nl}\ u_{nl}(r)\,.$$

Discuss the asymptotic behaviour of $u_{nl}(r)$ for $r \to 0$. [3 marks]

(c) For $l = 0$, find $R_{nl}(r)$ by solving the reduced radial equation in terms of trigonometric functions (there is no need to normalise $R_{nl}(r)$), and show that the eigenvalues of the energy are $E_{n0} = \frac{n^2\hbar^2\pi^2}{2ma^2}$ for $n \geq 1$. [6 marks]

(d) Obtain an order of magnitude for the energy of the ground state E_{10} by estimating p_r, Δp_r, and using $E_{10} = (\Delta p_r)^2/2m$. To do this, first compute $[\hat{p}_r, \hat{r}]$ recalling that the coordinate representation of $\hat{p}_r$ is $-i\hbar\frac{1}{r}\frac{d}{dr}r$, then use the generalised uncertainty relation for $\hat{p}_r$ and r to obtain an order of magnitude value for Δp_r. Finally, get an estimate for the energy. [4 marks]

3. Neutrinos can be described by the flavour states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ or by the massive states $|\nu_1\rangle$ and $|\nu_2\rangle$. The massive states are the eigenstates of the Hamiltonian $\hat{H}_0$, with eigenvalues $E_1 = \sqrt{p^2c^2 + m_1^2c^4} \simeq pc + \frac{m_1^2c^4}{2pc}$ and $E_2 = \sqrt{p^2c^2 + m_2^2c^4} \simeq pc + \frac{m_2^2c^4}{2pc}$. p is the common momentum, c the velocity of light, while m_1 and m_2 indicate the respective masses of $|\nu_1\rangle$ and $|\nu_2\rangle$, and both masses are much less than p/c. The two bases are related by a unitary transformation U, using the mixing angle θ:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}.$$

(a) Write the matrix representation of the Hamiltonian $\hat{H}_0$ in the massive basis, $|\nu_1\rangle$ and $|\nu_2\rangle$. [1 mark]

(b) Show that the matrix representation of the Hamiltonian $\hat{H}_0$ in the flavour basis, $|\nu_e\rangle$ and $|\nu_\mu\rangle$, is:

$$H_0|_{\nu_e,\nu_\mu} = \begin{pmatrix} E_1\cos^2\theta + E_2\sin^2\theta & (E_2 - E_1)\sin\theta\cos\theta \\ (E_2 - E_1)\sin\theta\cos\theta & E_1\sin^2\theta + E_2\cos^2\theta \end{pmatrix}.$$

[6 marks]

(c) Compute the eigenvalues of $H_0|_{\nu_e,\nu_\mu}$. Are they the same as the ones in the massive basis? [3 marks]

When neutrinos traverse the Earth, they interact very weakly with the electrons, protons and neutrons contained in the atoms. This effect is included in the Hamiltonian in matter, H_m, which, in the flavour basis, is given by:

$$H_\mathrm{m} = \begin{pmatrix} -\Delta m_{21}^2\cos(2\theta)\frac{c^4}{4pc} & \Delta m_{21}^2\sin(2\theta)\frac{c^4}{4pc} \\ \Delta m_{21}^2\sin(2\theta)\frac{c^4}{4pc} & \Delta m_{21}^2\cos(2\theta)\frac{c^4}{4pc} \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

A, real and > 0, depends on the density of electrons. $\Delta m_{21}^2 = m_2^2 - m_1^2$.

(d) Find $\tan(2\theta_m)$, θ_m being the mixing angle which relates the flavour basis to the eigenbase of H_m. [2 marks]

[Hint: $\tan(2\theta_m) = \frac{2H_{12}^\mathrm{m}}{H_{22}^\mathrm{m} - H_{11}^\mathrm{m}}$, where H_{ij}^m are the elements of H_m.]

(e) For small θ in vacuum, discuss how the mixing angle θ_m changes depending on A, for $\Delta m_{21}^2 > 0$. In particular, consider the cases of $A \ll \Delta m_{21}^2\cos(2\theta)\frac{c^4}{2pc}$, $A \gg \Delta m_{21}^2\cos(2\theta)\frac{c^4}{2pc}$ and $A = \Delta m_{21}^2\cos(2\theta)\frac{c^4}{2pc}$. [4 marks]

(f) The probability of ν_e oscillating into ν_μ at a time t in matter is given by $P(\nu_e \to \nu_\mu, t) = \sin^2(2\theta_m)\sin^2((E_1^m - E_2^m)t/2)$. E_1^m and E_2^m are the eigenvalues of H_{m}. Considering a sufficiently long time t such that $\sin^2((E_1^m - E_2^m)t/2) \sim 1$, and an experiment in which $A = 0.3\,\Delta m_{21}^2 \cos(2\theta)\frac{c^4}{2pc}$, is the probability of oscillation enhanced or suppressed for $\Delta m_{21}^2 > 0$, with respect to oscillations in vacuum? Similarly for $\Delta m_{21}^2 < 0$? [4 marks]

[Hint: Express $\sin^2(2\theta_m)$ as a function of $\tan^2(2\theta_m)$.]

4. The spin angular operator $\hat{\vec{S}}$ can be written in terms of the Pauli spin operators, $\hat{\vec{\sigma}}$, as: $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$.

(a) In the eigenbase of $\hat{S}_z$, denoted $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, write the matrix representations of $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ and $\hat{S}^2$. [4 marks]

[Hint: Recall that the Pauli matrices are given by $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and that they are the matrix representation of the Pauli operators in the eigenbase of $\hat{S}_z$.]

Consider a spin-1/2 particle which is in a state $|\gamma\rangle$:

$$|\gamma\rangle = a|+\rangle + b|-\rangle\ ,$$

with $|a|^2 + |b|^2 = 1$.

(b) Write $|\gamma\rangle$ as a column vector in the matrix representation in the eigenbase of $\hat{S}_z$. Compute the expectation values $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$ for the state $|\gamma\rangle$. [6 marks]

(c) What will be the respective probabilities that a measurement of $\hat{S}_z$ finds $\hbar/2$ and $-\hbar/2$ for $|\gamma\rangle$? [2 marks]

(d) What will be the probability that a measurement of $\hat{S}_x$ finds $\hbar/2$ for $|\gamma\rangle$? [4 marks]

[Hint: First find the eigenvalues and eigenvectors of $\hat{S}_x$.]

(e) Show that it is impossible for a spin 1/2 particle to be in a state $|\xi\rangle = c\,|+\rangle + d\,|-\rangle$ (normalised, i.e. $|c|^2 + |d|^2 = 1$) such that

$$\langle \hat{S}_x \rangle = 0, \qquad \langle \hat{S}_y \rangle = 0, \qquad \langle \hat{S}_z \rangle = 0\ .$$

[4 marks]

SECTION B. SPECIAL RELATIVITY AND ELECTROMAGNETISM
Answer question 5 and **at least one** of questions 6, 7 and 8.

5. (a) Muons have a rest mass of 106 MeV/c^2 and an average lifetime of 2.4×10^{-6} s. If a muon has an energy of 180 MeV, how far on average does it travel before it decays? [4 marks]

   (b) Find the matrix $\Lambda^{\mu}{}_{\nu}$ of a Lorentz transformation that is obtained by first boosting with velocity v along the x-axis and then rotating about the x-axis through an angle α. Does it matter in which order the boost and the rotation are performed? [4 marks]

   (c) Give the definition of the dual field-strength tensor $\widetilde{F}^{\mu\nu}$ in terms of the 4-potential. Use this definition to show $\partial_\mu \widetilde{F}^{\mu\nu} = 0$. [4 marks]

   (d) A certain 4-vector w^μ related to a particle of rest mass m and velocity $\underline{v}$ has spatial components $\underline{w} = 2c\gamma(v)\underline{v}/m$. Find w^0. [4 marks]

   (e) Two protons collide head-on to produce two protons and a pion, $p\,p \to p\,p\,\pi$. Find the minimal relative velocity of the incoming protons which allows this process to take place. [4 marks]

   [Hint: the rest mass of the proton and pion are given by $m_p = 940$ MeV/c^2 and $m_\pi = 140$ MeV/c^2 respectively.]

   (f) Consider a rod of proper length l_0. Show that there is no length contraction if an observer moves perpendicular to the direction of the rod. [4 marks]

   (g) Express the 0-component of the Maxwell equation $\partial_\mu F^{\mu\nu} = j^\nu/(c\,\epsilon_0)$ in terms of the electric and magnetic fields. [Hint: see question 6 for the definition of $F^{\mu\nu}$.] [4 marks]

   (h) The Lagrangian for a free, relativistic particle of rest mass m moving with velocity v is given by $\mathcal{L} = k\,c/\gamma(v)$. Use the non-relativistic limit to determine k. [4 marks]

6. We consider a point charge q of rest mass m in an electromagnetic field specified by the field-strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} .$$

The 4-force f^μ acting on the point charge is defined by

$$f^\mu \equiv \frac{dp^\mu}{d\tau} ,$$

where τ and p^μ are the proper time and 4-momentum of the point charge.

(a) Show that f^μ is a 4-vector and find its relation to the usual force $\underline{F} \equiv d\underline{p}/dt$. [4 marks]

(b) The 4-force acting on the point charge due to the electromagnetic field is given by

$$f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu ,$$

where u_ν is the 4-velocity of the point charge. Show that the spatial components of this equation correspond to the Lorentz force law. Derive and interpret the additional equation due to the 0-component of the above equation. [8 marks]

(c) The point charge is injected into a pure, constant magnetic field $\underline{B}$. Given that the initial velocity $\underline{v}$ of the point charge is perpendicular to $\underline{B}$ show that the point charge moves with constant speed on a circle. Compute the radius of the circle. What happens if the initial velocity is not perpendicular to $\underline{B}$? [8 marks]

7. Let S and S' be two inertial frames in standard configuration (S' is moving with velocity v along the x-axis and at $t = t' = 0$ the two frames coincide). The potential ϕ and vector potential $\underline{A}$ are the components of the 4-potential $A^\mu = (\phi, c\,\underline{A})$ and are related to the electric and magnetic fields $\underline{E}$ and $\underline{B}$ by

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad \text{and} \quad \underline{B} = \underline{\nabla} \times \underline{A} \; .$$

(a) An observer at rest in S measures a certain value for ϕ and $\underline{A}$ at some point in Minkowski space. Find ϕ' and $\underline{A}'$, the potential and vector potential at the same point, as seen by an observer at rest in S'. [4 marks]

(b) Given $\underline{E}$ and $\underline{B}$ as seen by an observer at rest in S, use the transformation property of A^μ to determine the z-component of the electric field and the x-component of the magnetic field as seen by an observer at rest in S'. [8 marks]

(c) An observer at rest in S sees a pure electric field, i.e. $\underline{B} = 0$, due to a point charge at rest. Find all inertial frames S'' where the corresponding electromagnetic field is also a pure electric field. [4 marks]

(d) Show that $\underline{E}$ and $\underline{B}$ are not affected by gauge transformations

$$\phi \to \phi - \frac{\partial \psi}{\partial t}, \qquad \underline{A} \to \underline{A} + \underline{\nabla}\psi \; ,$$

where $\psi(t, \underline{x})$ is an arbitrary function. Write the gauge transformations in covariant form. [4 marks]

8. Consider two particles with rest masses m_1 and m_2 just prior to a head-on collision in their centre-of-mass frame S. Their energies in S are given by E_1 and E_2 respectively and we denote their 4-momenta by $p_1^\mu = (E_1/c, \underline{p}_1)$ and $p_2^\mu = (E_2/c, \underline{p}_2)$. The invariant mass M of the pair is defined through $c^2\, M^2 \equiv (p_1 + p_2)^2$.

   Express the total centre-of-mass energy $E_1 + E_2$ in terms of Lorentz-invariant quantities. [4 marks]

   The laboratory frame S_L is the rest frame of particle 2. Compute the energy of particle 1 in S_L in terms of M, m_1 and m_2 and use this result to show that

$$\underline{p}_L^2 = \frac{c^2}{4m_2^2}\,[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]\,,$$

   where $\underline{p}_L$ is the momentum of particle 1 in S_L. [6 marks]

   Compute the relative velocity of S_L and S and express it in terms of E_2 and $\underline{p}_1$. [5 marks]

   At the HERA collider electrons with an energy of 30 GeV $= 3 \times 10^4$ MeV collide head on with protons of energy 820 GeV. Given that the rest masses of the electron and proton are $m_e = 0.5$ MeV$/c^2$ and $m_p = 940$ MeV$/c^2$, compute the invariant mass M of such an electron-proton pair. If the same invariant mass was to be obtained in a fixed-target experiment, i.e. with a proton at rest, to what energy would the electron have to be accelerated? [5 marks]